ARE UTILITY FUNCTIONS BOUNDED?

May 14, 1973

Arnold Faden

Let $A = \{a_1, a_2, \ldots\}$ be a sequence of possible outcomes one is interested in (e.g. income levels, votes, population killed in a war, unemployment, etc). A probability distribution over A may be represented by a sequence $p_1, p_2 \ldots$, where p_n is the probability of outcome a_n, ($p_n \geq 0$, all n, and $p_1 + p_2 + \ldots = 1$). The basic problem is to describe the structure of "rational" preferences among different probability distributions over A. Write $p \geq q$ if probability distribution p is preferred or indifferent to distribution q. Assume $\geq$ is <u>transitive</u> and <u>reflexive</u>, but not necessarily <u>complete</u> (i.e. there may be non-comparable distributions). $>$ stands for strict preference, $\sim$ for indifference.

A utility function $u = A \rightarrow$ reals may be represented by a sequence $u_1, u_2, \ldots$, where u_n is the utility of outcome a_n. For given distribution p, the expected utility is $p_1 u_1 + p_2 u_2 + \ldots$

Distribution p is preferred or indifferent to q on the <u>expected utility criterion</u> (notation $p \geq_u q$) iff $p_1 u_1 + p_2 u_2 + \ldots \geq q_1 u_1 + q_2 u_2 + \ldots$.

The trouble is that either or both of these series may be infinite or undefined if u is <u>unbounded</u>. To gain more comparability we <u>change</u> the definition to read.

$p \geq_u q$ iff $(p_1 - q_1) u_1 + (p_2 - q_2) u_2 + \ldots \geq 0$ (absolute convergence).

Say that utility function u <u>represents</u> the preference order $\geq$ iff $[p \geq_u q \longleftrightarrow p \geq q]$.

We now list some plausible axioms that a "rational" preference order should satisfy. First some definitions.

Distribution p is <u>finitely</u> <u>concentrated</u> iff $p_n = 0$ for in all but a finite number of indices n. A sequence $x_1, x_2, \ldots$ is <u>monotone</u> iff $x_1 \leq x_2 \leq \ldots$, or $x_1 \geq x_2 \geq \ldots$. Let P be the set of <u>all</u> probability distributions on A, and let F be the set of all <u>finitely-concentrated</u> distributions on A. $\in$ stands for set membership.

<u>Axiom 1</u> (finite comparability) Any two finitely-concentrated distributions are comparable ($p \geq q$ or $q \geq p$).

<u>Axiom 2</u> (strong independence) Let $p', q' \in F$, $p'', q'' \in P$, with $p' > q'$ and $p'' \sim q''$, and let $0 < t < 1$; then

$$[\, tp' + (1-t)\, p''\,] > [\, tq' + (1-t)\, q''\,].$$

<u>Axiom 3</u> For all $p', q' \in P$ with $p' > q'$, there exists $p'', q'' \in F$ and $0 < t < 1$ such that $q'' > p''$ and

$$[\, tp' + (1-t)\, p''\,] > [\, tq' + (1-t)\, q''\,].$$

<u>Axiom 4</u> (Archimedean) let $p, q \in P$, and let p_k, q_k, $k = 1,2,\ldots$, be sequences in F such that

$$\lim_{k \to \infty} p_k(a) = p(a) \quad , \quad \lim_{k \to \infty} q_k(a) = q(a)$$

for all outcomes $a \in A$; also let the three sequences $p_k(a)$, $q_k(a)$, $[p_k(a) - q_k(a)]$, $k = 1,2,\ldots$, be monotone for all outcomes $a \in A$; finally, let $p_k > q_k$ for all $k = 1, 2, \ldots$; then it is <u>false</u> that $q \succ p$.

<u>Axiom 5</u> (maximal comparability) Let $\geq'$ be another partial ordering on P satisfying axioms (1) through (4), and such that, if $p > q$ then $p >' q$, and, if $p \sim q$ then $p \sim' q$; then $\geq$ and $\geq'$ are identical.

<u>Fundamental Theorem</u>: A preference order $\geq$ satisfies axioms (1) through (5) if and only if there exists a utility function u (not necessarily bounded) which represents $\geq$.