Econ 565
Location Theory

Faden

Notes on Networks

I think that I shall never see
A network lovely as a tree...

A <u>graph</u> (or <u>network</u>) consists of <u>nodes</u>, and <u>arcs</u> between some pairs of nodes; arcs may be directed or undirected; a <u>path</u> is a sequence of arcs (ab), (bc), ... etc., successive arcs having a node in common; a <u>cycle</u> is a path which comes back to where it started; a graph is <u>connected</u> if any two nodes have a path from one to the other.

A <u>tree</u> is a connected graph without cycles; a tree in an n-node graph has n-1 arcs; there are n^{n-2} such trees; a <u>bipartite graph</u> is one in which the nodes are partitioned into two sets, A and B, such that all arcs are directed from A-nodes to B-nodes; if A has m nodes and B has n nodes there are $m^{n-1}\ n^{m-1}$ bipartite trees; an <u>arborescence</u> is a tree with all arcs directed away from a given node.

<u>Shortest spanning tree problem</u>; given n nodes with symmetric distances d_{ij} ($=d_{ji}$) of arc (i, j), find the tree the sum of whose distances is a minimum.

<u>Solution</u>: (J. Kruskal): connect the two closest nodes, then the two next closest, etc., unless such an arc would close a cycle; continue until n-1 arcs are selected.

<u>Shortest path problem</u>: given $d_{ij} \geq 0$ find the shortest (directed) path from specified node i_o to each other node..

<u>Solution</u> (G. Dantzig): construct an arborescence from i_o in n-1 steps; at the m-th step m nodes have been connected and assigned prices recursively by the rule: $P_{io} = 0$, $P_j = P_i + d_{ij}$; add arc (j, k) for which $P_j + d_{jk}$ is smallest, over nodes j already connected and k not already connected.

<u>Transshipment Problem</u>: given n nodes with net capacities c_i, and distances d_{ij}, choose X_{ij}, $i,j = 1 \ldots n$, to minimize $\Sigma_i \Sigma_j d_{ij} X_{ij}$, subject to $\Sigma_j (X_{ij} - X_{ji}) \leq t_i$, $i = 1, \ldots, n$, $X_{ij} \geq 0$ (this can be converted to equality constraints by adding a fictitious "storage" node to which surpluses are shipped). This is a linear program, and the basic feasible solutions are the <u>trees</u>.

<u>Solution</u>: one approach is to reduce to transportation problem by either (i) solving the shortest route problem for all pairs i,j with $c_i > 0 > c_j$ or (ii) (Orden) solving the n + n transportation problem $\Sigma\Sigma d_{ij} X_{ij}$ subject to $\Sigma_j X_{ij} = M + \max(c_i, 0)$, $\Sigma_i X_{ij} = M + \max(-c_i, 0)$, M a large number. ($d_{ii} = 0$, $\Sigma_i c_i = 0$).

<u>Transportation problem</u>: given m nodes with capacities $c_i > 0$ and n nodes with requirements $r_j > 0$ and distances (costs) d_{ij}, $i = 1 \ldots m$, $j = 1 \ldots n$, minimize $\Sigma_i \Sigma_j d_{ij} X_{ij}$ subject to $\Sigma_i X_{ij} \leq c_i$, $i = 1, \ldots m$,

$$\Sigma_j X_{ij} \geq r_j, \quad j = 1, \ldots n,$$

$X_{ij} \geq 0$ (This can be converted to equalities by adding a "storage" node with requirements $= \Sigma_i c_i - \Sigma_j r_j \geq 0$.) The basic feasible solutions are bipartite trees.

<u>Traveling Salesman problem:</u> given n nodes and distances d_{ij}, find the (directed) cycle through all nodes of minimal total distance.

<u>Solution</u>: only heuristic solution methods are known; if the metric is euclidean on the plane, no self-interesting cycle is optimal (M. Flood).

Maximum Flow problem: given n nodes with a specified origin 0 and destination D, capacities $c_{ij} \geq 0$ on directed arc (i,j), to maximize the total flow $\Sigma_j X_{oj} = \Sigma_j X_{jD}$, subject to, $0 \leq X_{ij} \leq c_{ij}$, all i,j, and $\Sigma_j X_{ij} = \Sigma_j X_{ji}$ on all nodes $i \neq 0$ or D.

A cut is a partition of the nodes into two sets, an original set A containing 0 and a destination set B containing D. The size of a cut is the sum of c_{ij} for all $i \epsilon A$, all $j \epsilon B$.

The min-cut-max-flow theorem states that the maximal flow equals the minimal size over all cuts (Ford and Fulkerson).