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7. THE TRANSPORTATION AND TRANSHIPMENT PROBLEMS

A 7.1. The Transportation Problem: Introduction[1]

The transportation problem with m sources and n sinks is: Find mn nonnegative numbers x_{ij}, ($i = 1,\ldots,m$; $j = 1,\ldots,n$), satisfying

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$$x_{i1} + x_{i2} + \ldots + x_{in} \leq \alpha_i, \qquad (7.1.1)\ \cancel{(1)}$$

($i = 1,\ldots,m$),

$$x_{1j} + x_{2j} + \ldots + x_{mj} \geq \beta_j, \qquad (7.1.2)\ \cancel{(2)}$$

($j = 1,\ldots,n$), and minimizing the sum of

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$$f_{ij}x_{ij} \qquad (7.1.3)\ \cancel{(3)}$$

over all mn terms of this form ($i = 1,\ldots,m$; $j = 1,\ldots,n$). Here the numbers α_i, β_j, and f_{ij} are given parameters (α_i, $\beta_j \geq 0$).

The most straightforward interpretation of this problem is the following: x_{ij} is the quantity of a certain commodity moving from a source i (say a manufacturing plant) to a sink j (say a market where the good is sold); f_{ij} is *unit* transport cost incurred by this movement, so that (3) is *total* transport cost for the source-sink pair (i,j). The problem, then, is to minimize the grand total of costs over all such pairs. Source i has a *capacity* α_i, and the constraints (1) state that the total shipments from a source cannot exceed its capacity. There are m such constraints, one for each source. Sink j has a requirement β_j, and the n constraints (2) state that total shipments into sink j must not fall below its requirement.

Besides this interpretation — (from which the transportation problem gets its name) — there are a remarkable number of others, concerning resource assignments, scheduling, etc.[2]

Now consider the following problem involving measures. We are given two measure spaces, (A, Σ', μ') and (B, Σ'', μ''). (A, Σ') will be called the source space, and (B, Σ'') the sink space. measure μ' will be called the capacity measure, and μ'' the requirement measure. We assume throughout this chapter that μ' and μ'' are ~~sigma~~ σ-finite. We are also given a cost function $f: A \times B \to$ reals, assumed measurable with respect to the product ~~sigma~~ σ-field, $\Sigma' \times \Sigma''$ on $A \times B$.

The problem is to find a measure λ on the space $(A \times B, \Sigma' \times \Sigma'')$ satisfying

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$$\lambda(E \times B) \leq \mu'(E), \qquad (7.1.4)\ \cancel{(4)}$$

for all $E \in \Sigma'$,

$$\lambda(A \times F) \geq \mu''(F), \qquad (7.1.5)\ \cancel{(5)}$$

for all $F \in \Sigma''$, and minimizing

$$\int f \, d\lambda. \qquad (7.1.6)\ \cancel{(6)}$$

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Here (6) is an indefinite integral over space $\underline{A} \times \underline{B}$, and "minimization" is to be understood in the sense of (reverse) standard ordering of pseudomeasures. Of course, if the definite integral

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$$\int_{\underline{A}\times\underline{B}} \underline{f}\, d\lambda \qquad \text{(7.1.7)} \quad \cancel{(7)}$$

is well-defined and finite for all feasible λ, ~~then~~ this reduces to the ordinary minimization of (7). But there is no *a priori* guarantee that (7) *will* be finite, or even well-defined, without special conditions on $\underline{f}$, μ', and μ''.

Formulas (4), (5), and (6) reduce to (1), (2), and (3), respectively, iff both ~~sigma~~-fields, Σ' and Σ'', are finite. To be precise, let Σ' be generated by a partition $\{\underline{A}_1,\ldots,\underline{A}_m\}$ of $\underline{A}$, and Σ'' by a partition $\{\underline{B}_1,\ldots,\underline{B}_n\}$ of $\underline{B}$. Then it is a simple exercise to verify the preceding statement: $\mu'(\underline{A}_i) = \alpha_i$, $\mu''(\underline{B}_j) = \beta_j$, etc. This shows that we *are* dealing with a *bona fide* generalization of the ordinary transportation problem.

The interpretations ~~which~~ that can be given to (4)~~, (5), (6)~~ include all those for the ordinary "discrete" problem, and the greater flexibility ~~which~~ that one attains with measures enables one to fit the real situation that much more closely. For example, in the transportation interpretation, one may now treat the case where sources are spread more-or-less continuously over the surface of the earth (as in agricultural production), ~~or~~ where sinks are — as in the sale of consumer goods to a diffused population — or both. The best the ordinary problem (1)-(3) ~~(1), (2), (3)~~ can do is to aggregate these distributions; e.g. ~~for example,~~ to treat countries as if located at single points in international trade models.

Again, in some interpretations the sources and sinks correspond to time-instants rather than locations; here the measure-theoretic formulation allows us ~~one~~ to work with continuous time, rather than having to lump things into discrete periods. This has clear theoretical advantages; it may even be advantageous in practical applications, since "continuous" models are often simpler than "discrete" models.

One of the most important non-transport interpretations of the transportation problem refers to the assignment of resources to activities. Here the "sources" are the various kinds of resources available, and the "sinks" are the various activities. The measure-theoretic generalization is especially welcome in this interpretation, to allow for the infinite variety of resources and activities. ~~In fact,~~ In ~~the next~~ chapter 8 we show that Thünen systems can be represented by just such a model.

We ~~Let us~~ now examine (4)-(6) ~~(4), (5), (6)~~ more closely. $\mu'(E)$ gives the total capacity of the set of sources E, while $\lambda(E \times B)$ gives the total outflow from these sources. (4) then is the condition that outflow not exceed capacity for any measurable set of sources. A similar relation, (5), holds between $\mu''(\underline{F})$, the requirement for the set of sinks $\underline{F}$, and $\lambda(\underline{A} \times \underline{F})$, the total inflow into these sinks.

It will be convenient to formulate the constraints in terms of marginals. Recall that the <u>left</u> <u>marginal</u> of $(\underline{A} \times \underline{B}, \Sigma' \times \Sigma'', \lambda)$ is the measure λ' on $(\underline{A}, \Sigma')$, ~~which is~~ given by

$$\lambda'(\underline{E}) = \lambda(\underline{E} \times \underline{B})$$

for all $\underline{E} \in \Sigma'$. Similarly, the <u>right</u> <u>marginal</u> is the measure λ'' on $(\underline{B}, \Sigma'')$, given by

$$\lambda''(\underline{F}) = \lambda(\underline{A} \times \underline{F})$$

for all $\underline{F} \in \Sigma''$. It follows at once that the constraints (4), (5) can be written in the very simple form:

$$\lambda' \leq \mu' \qquad (7.1.8)\ ~~(8)~~$$

$$\lambda'' \geq \mu'' \qquad (7.1.9)\ ~~(9)~~$$

respectively. ~~Let~~ We us refer to λ' and λ'' as the <u>outflow</u> and <u>inflow</u> measures, respectively, and to λ itself as the <u>flow</u> <u>measure</u>.

Now we discuss ~~for~~ a point ~~which~~ that was glossed over. For (6) to be well-defined as a pseudomeasure, λ must be ~~sigma~~ σ-finite. Is this guaranteed? We examine the situation in some generality, because it recurs several times in this chapter.

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Let $(\underline{A}_i, \Sigma_i, \mu_i)$, $\underline{i} = 1,2$, be two measure spaces, and $\underline{g}:\underline{A}_1 \to \underline{A}_2$ measurable, such that the following relation obtains:

(7.1.70)

$$\mu_2(\underline{E}) = \mu_1\{\underline{a}_1 | \underline{g}(\underline{a}_1) \in \underline{E}\}, \tag{10}$$

for all $\underline{E} \in \Sigma_2$. ~~That is~~ i.e., μ_2 is the measure induced from μ_1 by $\underline{g}$.

Now, if μ_2 is ~~sigma~~ σ-finite, then μ_1 is ~~sigma~~ σ-finite. To show this, let $\bar{G}$ be a countable measurable partition of $\underline{A}_2$ such that $\mu_2(\underline{G})$ is finite for all $\underline{G} \in \bar{G}$. The collection of sets $\{\underline{a}_1 | \underline{g}(\underline{a}_1) \in \underline{G}\}$, $\underline{G} \in \bar{G}$, is then a countable measurable partition of $\underline{A}_1$ and, by ~~(11)~~ (10), μ_1 is finite on each of these sets. Hence μ_1 is ~~sigma~~ σ-finite if μ_2 is. This completes the proof.

18 The converse of this statement is not necessarily true. As an example, let μ_1 be any infinite ~~sigma~~ σ-finite measure, and let $\underline{A}_2$ consist of a single point. Then $\mu_2(\underline{A}_2) = \infty$, and this is clearly not ~~sigma~~ σ-finite.[3]

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Now consider the transportation problem. The left marginal, λ', is the measure induced from λ by the projection $\underline{g}(\underline{a},\underline{b}) = \underline{a}$. Hence, if λ' is ~~sigma~~ σ-finite, so is λ. But we are given that ~~χ''~~ μ' is ~~sigma~~ σ-finite, and it follows from ~~(7)~~ (8) that λ' is ~~sigma~~ σ-finite. We conclude that, ~~indeed,~~ any feasible flow λ is ~~sigma~~ σ-finite, and (6) is well-defined. Note, however, that the right marginal, λ'', is not necessarily ~~sigma~~ σ-finite.

μ prime

It is common practice in analyzing the transportation problem to replace some of the inequality signs in (1) and (2) by equalities. We ~~shall~~ also consider the consequences of replacing one or both of the inequalities (8) (9) by equalities. This gives altogether four variants of the transportation problem. We ~~shall~~ label these types I, II, III, and IV, defined as follows:

	$\lambda'' = \mu''$	$\lambda'' \geq \mu''$
$\lambda' = \mu'$	I	III
$\lambda' \leq \mu'$	II	IV

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Thus in types I and II, requirements must be met exactly, without oversupply. In types I and III capacity must be fully utilized. Type IV is our original problem, given by (8) and (9). In all four variants the objective remains the same: to minimize (6).

The parallel analysis of these four types is quite instructive, and they exhibit a surprising degree of individuality.

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7.2. The Transportation Problem: Existence of Feasible Solutions

The first problem we tackle is this: under what conditions does ~~there~~ exist a flow measure λ satisfying the feasibility conditions (1.8) ~~(10)~~ and (1.9) ~~(11)~~, or their equality-constrained counterparts? The objective function (1.6) ~~(6)~~ plays no part in this discussion.

For the ordinary transportation problem we have the following well-known results: (1.1) ~~(1)~~ and (1.2) ~~of section 1~~ have a feasible solution iff total capacity is at least as large as total requirements, i.e. ~~That is~~, iff

$$\alpha_1 + \ldots + \alpha_m \geq \beta_1 + \ldots + \beta_n \qquad (7.2.1)\ \cancel{(1)}$$

Furthermore, this remains true in the case where all inequality signs in (1.1) ~~(1) of section 1~~, or in (1.2) ~~(2) of section 1~~ (but not both) are replaced by equalities. Finally, if all constraints are equalities, ~~then~~ a solution exists iff (1) is satisfied with equality. Thus ~~In other words,~~ (1) is necessary and sufficient for the existence of feasible solutions in variants II, III and IV; and (1) with equality in variant I.[4]

Our main feasibility result is that these conditions carry over completely to the measure-theoretic transportation problem. The demonstration of this is by no means trivial, especially when the capacity and requirement measures are infinite.

Our first result establishes feasibility for the ordinary transportation problem (variant I) extended to the case where the number of sources and sinks is countable.

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Lemma: Let $\alpha_1, \alpha_2, \ldots,$ and $\beta_1, \beta_2, \ldots$ be two sequences of non-negative real numbers, such that

$$\alpha_1 + \alpha_2 + \ldots = \beta_1 + \beta_2 + \ldots \qquad (7.2.2)\ \cancel{(2)}$$

Then ~~there~~ exist non-negative numbers x_{ij} satisfying

$$x_{i1} + x_{i2} + \ldots = \alpha_i \qquad (7.2.3)\ \cancel{(3)}$$

for all α_i, and

$$x_{1j} + x_{2j} + \ldots = \beta_j \tag{7.2.4}$$

for all β_j.

(The two sequences may be finite or infinite, and need not be equal in length; the common sum in (2) may be finite or infinite; i indexes the sequence (α_i), j the sequence (β_j)).

subscript zero where inked

Proof: Define $m_i = \alpha_1 + \ldots + \alpha_i$, $n_j = \beta_1 + \ldots + \beta_j$, $m_0 = n_0 = 0$; ~~and~~ then let (for $i = 1, 2, \ldots, j = 1, 2, \ldots$)

$$x_{ij} = \min(m_i, n_j) - \min(m_i, n_{j-1})$$
$$- \min(m_{i-1}, n_j) + \min(m_{i-1}, n_{j-1}). \tag{7.2.5}$$

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We show that (5) gives the desired feasible solution. First, we show that all the numbers x_{ij} are non-negative.

Suppose that $m_{i-1} \leq n_{j-1}$. Then the last two terms on the right of (5) cancel, and the difference of the first two is clearly non-negative. (Remember that $n_j \geq n_{j-1}$). A similar argument obtains if $m_{i-1} \geq n_{j-1}$. Hence x_{ij} is non-negative.

Next, we verify that, for all indices j of the sequence (β_j), and all indices i of the sequence (α_i), we have

$$x_{i1} + x_{i2} + \ldots + x_{ij} = \min(m_i, n_j) - \min(m_{i-1}, n_j). \tag{7.2.6}$$

Proceed by induction on j. For $j = 1$, (6) follows immediately from (5), since $n_0 = 0$. Suppose ~~(6)~~ (6) is true for $j - 1$ in place of j, we add (5) to it, and obtain (6) *per se*. Hence (6) is true in general.

Now in (6) let j increase indefinitely (if (β_j) is an infinite sequence) or to its maximum value (if (β_j) is a finite sequence). In either case we find that $\lim n_j \geq m_i$, because of (2). But this means that in the limit the right side of (6) simplifies to $m_i - m_{i-1} = \alpha_i$. Thus (3) is verified. The same argument with i and j interchanged verifies (4).

x_{ij} as given by (5) has a very simple interpretation: it is precisely the "northwest corner" solution for the ordinary transportation problem.[5] Specifically, one starts by making x_{11} as large as possible without automatically violating (3) or (4) ~~- that is~~ i.e., take $x_{11} = \min(\alpha_1, \beta_1)$. If $x_{11} = \alpha_1$, then all the other x_{1j} must be set equal to zero to satisfy (3) for $i = 1$; similarly, if $x_{11} = \beta_1$, all the other $x_{i1} = 0$. Now go to the as-yet-undetermined x_{ij} for which $i + j$ is as small as possible, and make *it* as large as possible, subject to not automatically violating (3) or (4). This recursive procedure yields (5). ~~What~~ we have ~~done is to~~ show that it still yields a feasible solution even for a *countable* number of sources and sinks, provided (2) holds.

The northwest corner solution will play an important rôle in ~~the next~~ chapter 8. In fact, for Thünen systems a certain (generalized) northwest corner solution is not only feasible but *optimal*, and encapsulates in a striking way the main structural features of such systems.

We now come to the main result. Essential use is made of the product measure theorem. Recall that, if $(\underline{A},\Sigma',\mu')$ and $(\underline{B},\Sigma'',\mu'')$ are measure spaces with μ, ν ~~sigma~~ σ-finite, or even ~~abcont,~~ arbitrary, ~~then there~~ exists a measure λ on the product space $(\underline{A} \times \underline{B},\ \Sigma' \times \Sigma'')$ with the property that $\lambda(\underline{E} \times \underline{F}) = \mu'(\underline{E})\mu''(\underline{F})$ for all $\underline{E} \in \Sigma'$, $\underline{F} \in \Sigma''$. This is called ~~the~~ a product measure and denoted $\mu' \times \mu''$.

Theorem: Let $(\underline{A},\Sigma',\mu')$ and $(\underline{B},\Sigma'',\mu'')$ be ~~sigma~~ σ-finite measure spaces ~~which~~ that are the source and sink spaces, respectively, of a transportation problem; let $\underline{B} \neq \emptyset$. ~~There~~ exists a feasible flow measure λ for this problem iff

$$\mu'(\underline{A}) = \mu''(\underline{B}) \qquad \text{~~(7)~~ (7.2.7)}$$

in variant I, and iff

$$\mu'(\underline{A}) \geq \mu''(\underline{B}) \qquad \text{~~(8)~~ (7.2.8)}$$

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in variants II, III, and IV.

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Proof: The "only if" part is simple to demonstrate. Letting λ be feasible, we find that

$$\mu'(\underline{A}) \geq \lambda(\underline{A} \times \underline{B}) \geq \mu''(\underline{B})$$

in variants II, III, IV, while the same holds with equalities in variant I.

18 Now ~~for~~ we demonstrate the "if" part. ~~First of all,~~ any λ feasible for II or III is also feasible for IV. Hence it suffices to prove the existence of feasible λ for types II and III only, under

assumption (8) (as well as feasibility for type I under (7), ~~of course).~~

We consider three cases, depending on the magnitude of $\mu''(B)$.

case (i): $\mu''(B) = 0$.

For variant I, (7) implies that $\mu'(A) = 0$ also. Hence the identically zero measure on $(A \times B, \Sigma' \times \Sigma'')$ is feasible. $\lambda = 0$ is also feasible for II, since only requirements must be satisfied with equality. As for III, choose an arbitrary point $b_o \in B$, and define λ as follows:

$$\lambda(G) = \mu'\{a \mid (a, b_o) \in G\}$$

for all $G \in \Sigma' \times \Sigma''$. This λ is well-defined, because "cross-sections" of measurable sets are measurable. It is easily verified to be a measure, and $\lambda(E \times B) = \mu'(E)$ for all $E \in \Sigma'$. Hence the conditions for transportation variant III are satisfied. ~~This completes case (i).~~

case (ii): $\infty > \mu''(B) > 0$.

For variants I and III, take λ proportional to the product measure:

$$\lambda = \frac{\mu' \times \mu''}{\mu''(B)}.$$

We then have $\lambda(E \times B) = \mu'(E)$, while $\lambda(A \times F)$ $(=, \geq)$ $\mu''(F)$ for variants (I, III), because of (7) and (8), respectively. Hence λ is feasible.

Variant II is slightly more complicated. If $\mu'(A) = \infty$, ~~then,~~ because μ' is ~~sigma~~ σ-finite, hence abcont, ~~there exists~~ a measure $\tilde{\mu}$ exists on A such that $\tilde{\mu} \leq \mu'$ and

$$\infty > \tilde{\mu}(A) \geq \mu''(B). \qquad \text{(7.2.9)} \; \cancel{(9)}$$

(If $\mu'(A)$ is finite, we simply take $\tilde{\mu} = \mu'$; (9) then follows from (8)). Now define λ by

$(\tilde{\mu} \times \mu'')/\tilde{\mu}(A)$

$$\lambda = \frac{\tilde{\mu} \times \mu''}{\tilde{\mu}(A)}.$$

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Then $\lambda(A \times F) = \mu''(F)$, all $F \in \Sigma''$. Also,

$$\lambda(E \times B) = \frac{\tilde{\mu}(E)\mu''(B)}{\tilde{\mu}(A)} \leq \tilde{\mu}(E) \leq \mu'(E),$$

all $E \in \Sigma'$.

Hence the conditions for variant II are satisfied. ~~This completes case (ii).~~

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18 case (iii): $\mu''(B) = \infty$.

This is the hard part. ~~First of all,~~ from (7) or (8) we obtain $\mu'(A) = \infty$ also. Hence it suffices to find a feasible λ for variant I, because this λ will also be feasible for II, III, and IV.

Since μ' is ~~sigma~~ σ-finite, hence abcont, it may be written as a countable sum of finite non-zero measures:

$$\mu' = \mu_1' + \mu_2' + \ldots$$

Similarly, for μ'' we may write

$$\mu'' = \mu''_1 + \mu''_2 + \ldots,$$

all μ''_j finite, non-zero. Define the sequences (α_i), $i = 1, 2, \ldots$, (β_j), $j = 1, 2, \ldots$, by

$$\alpha_i = \mu'_i(A), \quad \beta_j = \mu''_j(B).$$

These are positive real numbers.

¶ Now invoke the preceding lemma. (2) is satisfied, both sides summing to $+\infty$. Hence ~~there~~ exist non-negative real x_{ij}, $i, j = 1, 2, \ldots,$ satisfying (3) and (4).

Define λ as the sum of the product measures

$(\mu'_i \times \mu''_j)\, x_{ij} / (\alpha_i \beta_j)$

$$\left(\frac{x_{ij}}{\alpha_i \beta_j}\right) \cdot (\mu'_i \times \mu''_j),$$

the summation extending over all pairs (i, j), $i, j = 1, 2, \ldots$. We show that λ is feasible for variant I. For $E \in \Sigma'$, $\lambda(E \times B)$ is the sum of all terms of the form

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$$\left(\frac{x_{ij}}{\alpha_i \beta_j}\right) \mu'_i(E)\, \mu''_j(B) = \left(\frac{x_{ij}}{\alpha_i}\right) \mu'_i(E).$$

$\mu'_i(E)\, \mu''_j(B)\, x_{ij} / (\alpha_i \beta_j) = \mu'_i(E)\, x_{ij} / \alpha_i$

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Summing first over j, and using (3), we obtain $\mu'_i(E)$. Summing this over i, we obtain $\mu'(E)$. Hence the capacity constraint is satisfied: $\lambda(E \times B) = \mu'(E)$. A similar argument with i and j interchanged shows that $\lambda(A \times F) = \mu''(F)$, all $F \in \Sigma''$. Hence λ is feasible for variant I, hence feasible for II, III, and IV. This completes case (iii) and the proof. ||| □

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(Actually, ~~one~~ we see~~s~~ that this proof is valid even if μ' and μ'' are not ~~sigma~~ σ-finite, but merely abcont. In this case the λ constructed in the proof is also abcont.)

7.3. The Transportation Problem: Duality

Linear programs come in pairs, each being the dual of the other. The dual of the ordinary transportation problem, (1.1) $\frac{1}{N}$ (1.3) ~~(1), (2), (3) of section 1~~, is:

Find $m + n$ non-negative numbers $p_1,\ldots,p_m$, $q_1,\ldots,q_n$ satisfying

$$q_j - p_i \le f_{ij} \tag{7.3.1}$$

($i = 1,\ldots,m$; $j = 1,\ldots,n$), and maximizing

$$\beta_1 q_1 + \ldots + \beta_n q_n - \alpha_1 p_1 - \ldots - \alpha_m p_m. \tag{7.3.2}$$

This pair of programs has the following properties, shared by any pair of dual programs. If any feasible solutions are substituted in their respective objective functions, (1.3) ~~(3) of section 1~~, and (2), ~~then~~ the value of the minimizing objective, (1.3) ~~(3) of section 1~~, never falls below the value of the maximizing objective, (2) ~~above~~. ~~In fact,~~ a pair of feasible solutions are jointly optimal for their respective problems iff the values they impart to the objective functions are equal.

~~Furthermore,~~ this equality obtains iff these feasible solutions satisfy the following "complementary slackness" conditions. There is~~, first of all,~~ a natural 1-1 correspondence between the constraints of one problem and the variables of its dual. For the pair above, the constraints (1.1) ~~(1) of section 1~~ correspond to the variables p_i ($i = 1,\ldots,m$); (1.2) ~~(2) of section 1~~ corresponds to q_j ($j = 1,\ldots,n$); and (1) ~~above~~ corresponds to x_{ij} ($i = 1,\ldots,m$; $j = 1,\ldots,n$). The complementary slackness condition then states that, if a variable is positive, its corresponding constraint is satisfied with equality.

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The question now arises: Does the duality construction carry over to the measure-theoretic transportation problem, and does the resulting pair have properties analogous to those just mentioned? The answer is yes, up to a point.

Consider the transportation problem determined by the pair of ~~sigma~~ σ-finite measure spaces (A,Σ',μ'), (B,Σ'',μ''), with the constraints $\lambda' \le \mu'$ and $\lambda'' \ge \mu''$ [(1.8) and (1.9)] ~~(8) and (9) of section 1)~~ and the objective of minimizing $\int f\,d\lambda$ [(1.6)] ~~(6) of section 1)~~ over feasible flow measures λ ($f: A \times B \to$ reals is measurable).

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18 We define the dual of this problem as follows:

Find nonnegative measurable functions $p: A \to$ reals and $q: B \to$ reals ~~which~~ that satisfy

$$q(b) - p(a) \le f(a,b) \qquad (7.3.3)\ \text{~~(3)~~}$$

for all $a \in A$, $b \in B$, ~~and~~ for which the definite integrals

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$$\int_A p\,d\mu', \quad \int_B q\,d\mu'' \qquad (7.3.4)\ \text{~~(4)~~}$$

are both well-defined and finite,
and which maximize

$$\int_B q \, d\mu'' - \int_A p \, d\mu' \qquad \text{(7.3.5)} \; \cancel{(5)}$$

over all pairs (p,q) satisfying (3) and (4).

Let us ~~first~~ compare this with the ordinary dual, (1) and (2). We have already noted that the measure-theoretic reduces to the ordinary transportation problem exactly when the ~~sigma~~ 6-fields, Σ' and Σ'', are both finite. The same is true for the duals. More precisely, the situation is as follows. Let Σ', Σ'' be generated by the partitions $\{A_1, \ldots, A_m\}$, $\{B_1, \ldots, B_n\}$, respectively. Since p is measurable, it must be constant on each set A_i; let p_i be this value on A_i. Similarly, q has a constant value q_j on B_j, and f has a constant value f_{ij} on $A_i \times B_j$. Then (3) and (5) reduce to (1) and (2), respectively.

Condition (4) has no explicit counterpart in the ordinary dual; but, since the integrals reduce to finite sums, it is ~~of course~~ automatically satisfied in the ordinary dual. One might ask, however, why ~~condition~~ (4) should be included. Would not (3) and (5) alone be an adequate generalization of the ordinary dual?

One difficulty that arises if (4) is dropped is that the objective function (5) might no longer be well-defined for all feasible pairs (p,q). This difficulty is easily surmounted, as follows. First, assume that A and B are disjoint (this involves no real loss of generality, since common points can be formally distinguished). Next, interpret the expressions in (4) as indefinite integrals, in the sense of pseudomeasures. Finally, interpret (5) as the direct sum:

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$$\int q\, d\mu'' \oplus \int (-p)\, d\mu' . \qquad (7.3.6)\ \cancel{(6)}$$

(6) is then a pseudomeasure over $A \cup B$, and "maximization" is to be understood in the sense of standard order. If now (4) also holds, then everything is finite, and the standard ordering of (6) reduces to the ordinary size ordering of the definite integrals (5). Thus a perfectly reasonable problem results even if (4) is dropped.

Our main reason for inserting (4) is that with its aid we can prove that many ordinary duality properties generalize to the measure-theoretic case, whereas without it we can prove less. We ~~shall~~ refer to (3) and (5) alone, without (4), as the dual in the wide sense.

The entire discussion of duality to this point has been framed in terms of the inequality-constrained transportation problem — ~~that is~~ i.e., variant IV. For the other three variants, we define the dual exactly as above — ~~(3), (4) and (5)~~ (3) & (5) — but relax the non-negativity constraints on p and/or q. Specifically, if requirements must be met exactly (variants I and II), then q is allowed to take on negative values. And if capacity must be utilized fully (variants I and III), then p is allowed to take on negative values. This is completely analogous to what happens for the corresponding variants of the ordinary transportation problem, and indeed for dual linear programs in general. Equality constraints correspond to dual variables without sign restrictions.

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With these definitions, the following theorems hold for all four variants of the transportation problem, each with its particular dual. The measure-theoretic transportation problem (in any variant) will be called the <u>primal</u>. This and its dual are determined by the ~~sigma~~ σ-finite source and sink spaces, $(\underline{A},\Sigma',\mu')$ and $(\underline{B},\Sigma'',\mu'')$, and by the measurable cost function $\underline{f}:\underline{A} \times \underline{B} \rightarrow$ reals.

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<u>Theorem</u>: Let flow measure λ be feasible for the primal, and functions $(\underline{p},\underline{q})$ be feasible for the dual. Then $\int_{\underline{A}\times\underline{B}} \underline{f}\, d\lambda$ is well-defined, and

$$\int_{\underline{A}\times\underline{B}} \underline{f}\, d\lambda \geq \int_{\underline{B}} \underline{q}\, d\mu'' - \int_{\underline{A}} \underline{p}\, d\mu' . \qquad (7.3.7)\ \cancel{(7)}$$

D

<u>Proof</u>: We have

$$\int_{\underline{A}} \underline{p}\, d\mu' \geq \int_{\underline{A}} \underline{p}\, d\lambda' < \infty . \qquad (7.3.8)\ \cancel{(8)}$$

To show this, we consider two cases. In problem variants I and III, $\mu' = \lambda'$, and (8) is trivial; in variants II and IV, $\mu' \geq \lambda'$ and $\underline{p} \geq 0$, so (8) again is valid. (The right inequality in (8) follows from the left by (4).) Similarly,

$$\int_{\underline{B}} \underline{q}\, d\mu'' \leq \int_{\underline{B}} \underline{q}\, d\lambda'' > -\infty. \qquad (7.3.9)\ \cancel{(9)}$$

For, $\mu'' = \lambda''$ in variants $\underline{I}$ and $\underline{II}$, while $\mu'' \leq \lambda''$ and $\underline{q} \geq 0$ in variants III and IV. The right inequality in (9) again follows from the left.

From (8) and (9) we obtain

$$\int_{\underline{B}} \underline{q}\, d\mu'' - \int_{\underline{A}} \underline{p}\, d\mu' \leq \int_{\underline{B}} \underline{q}\, d\lambda'' - \int_{\underline{A}} \underline{p}\, d\lambda' > -\infty. \qquad (7.3.10)\ \cancel{(10)}$$

Now define the functions $\underline{p}_1$, $\underline{q}_1 : \underline{A} \times \underline{B} \to$ reals by: $\underline{p}_1(\underline{a},\underline{b}) = \underline{p}(\underline{a})$, and $\underline{q}_1(\underline{a},\underline{b}) = \underline{q}(\underline{b})$, all $\underline{a} \in \underline{A}$, $\underline{b} \in \underline{B}$. We find that

$$\int_{\underline{A}} \underline{p}\, d\lambda' = \int_{\underline{A}\times\underline{B}} \underline{p}_1\, d\lambda, \ \cancel{\text{and}}\ \int_{\underline{B}} \underline{q}\, d\lambda'' = \int_{\underline{A}\times\underline{B}} \underline{q}_1\, d\lambda, \qquad (7.3.11)\ \cancel{(11)}$$

by the induced integrals theorem. (Verify this separately for $\underline{p}^+$ and $\underline{p}_1^{\,+}$, and for $\underline{p}^-$ and $\underline{p}_1^{\,-}$, and subtract; similarly for $\underline{q}$, $\underline{q}_1$). From (10) and (11) we obtain

$$\int_{\underline{B}} \underline{q}\, d\lambda'' - \int_{\underline{A}} \underline{p}\, d\lambda' = \int_{\underline{A}\times\underline{B}} (\underline{q}_1-\underline{p}_1)\, d\lambda > -\infty. \qquad (7.3.12)\ \cancel{(12)}$$

Since $\underline{q}_1 - \underline{p}_1 \leq \underline{f}$ from (3), it follows that $\int_{\underline{A}\times\underline{B}} \underline{f}\, d\lambda$ is well-defined, for

$$\int_{\underline{A}\times\underline{B}} \underline{f}^-\, d\lambda \leq \int_{\underline{A}\times\underline{B}} (\underline{q}_1-\underline{p}_1)^-\, d\lambda < \infty,$$

from (12). And, in fact, from (3) we obtain

$$\int_{A\times B}(q_1-p_1)\,d\lambda \le \int_{A\times B} f\,d\lambda. \qquad \cancel{(13)}\ (7.3.13)$$

Finally, (10), (12), and (13) together yield (7). ▭

This theorem generalizes the ordinary duality property that a feasible value for the maximizing problem never exceeds a feasible value for the minimizing problem. Note that the mere existence of a dual-feasible pair (p,q) implies a condition on every feasible λ: that $\int_{A\times B} f\,d\lambda$ be well-defined, or, ~~in other words~~, that the indefinite integral $\int f\,d\lambda$ be a signed measure, not a proper pseudomeasure.

We now introduce a concept ~~which~~ that generalizes the notion of "complementary slackness."

Definition: Let flow measure λ on $(A \times B, \Sigma' \times \Sigma'')$, and the pair of functions $p{:}A \to$ reals, $q{:}B \to$ reals be feasible for the transportation problem and its dual, respectively. Pair (p,q) is a measure potential for λ iff the following ~~three~~ conditions are satisfied:

(i) $$\lambda\{(a,b)\,|\,q(b) - p(a) < f(a,b)\} = 0; \qquad \cancel{(14)}\ (7.3.14)$$

(ii) λ' and μ' coincide on subsets of $\{a\,|\,p(a) > 0\}$; ~~(15)~~ (7.3.15)

(iii) λ'' and μ'' coincide on subsets of $\{b\,|\,q(b) > 0\}$. ~~(16)~~ (7.3.16)

Condition (14) states that no flow occurs on the set where (3) is satisfied with strict inequality. Condition (15) states that capacity is fully utilized on the set of sources where p is positive. Condition (16) states that requirements are met exactly on the set of sinks where q is positive.

This definition is meant to apply to all four variants of the transportation problem. Note, however, that in variants I and III we have $\lambda' = \mu'$, so that (15) is automatically satisfied and may be dropped without changing the definition. Similarly, in variants I and II we have $\lambda'' = \mu''$, so that (16) must be true and may be dropped.

As an exercise, the reader might verify that this reduces to the ordinary "complementary slackness" conditions when both ~~sigma~~ σ-fields, Σ' and Σ'', are finite.

Note that dual feasibility of (p,q) (i.e. ~~that is~~, satisfaction of conditions (3) and (4)) is a requirement for measure potentiality. If as suggested above, (4) is dropped as a dual constraint, this gives rise to a correspondingly weaker concept here, which may be called measure potentiality in the wide sense. For the time being, however, we stick with the original concept.

D

Theorem: Let λ^{o} be feasible for the transportation problem, and (p^{o},q^{o}) feasible for its dual (given by (3) and (4)). Then (p^{o},q^{o}) is a measure potential for λ^{o} iff

$$\int_{A\times B} f\, d\lambda^{o} = \int_{B} q^{o}\, d\mu'' - \int_{A} p^{o}\, d\mu' . \qquad \text{(7.3.17)} \; \cancel{(17)}$$

Proof: Let (p°, q°) be a measure potential for λ°. Reviewing the preceding proof, we find that at (8), (9), and (13) the weak inequalities become equalities, because of (15), (16), and (14), respectively. Hence (7) is satisfied with equality; this is (17).

Conversely, let (17) be true. Then all the integrals appearing in the preceding proof must be finite, and the weak inequalities of (8), (9), and (13) must all be equalities. But finite equality in (8) implies condition (15). This is trivial in variants I and III; in variants II and IV it follows from the facts: $p^\circ \geq 0$, $\lambda^{\circ\prime} \leq \mu'$. Similarly, finite equality in (9) and (13) implies conditions (16) and (14), respectively. Hence (p°, q°) is a measure potential for λ°.

Theorem: If the pair (p°, q°) is a measure potential for λ°, then λ° is a best solution for the transportation problem, and (p°, q°) is a best solution for its dual ("best" in the sense of standard order).

Proof: Let λ and (p, q) be any other solutions feasible for the transportation problem and its dual, respectively. From the two preceding theorems we obtain

break at = sign if necessary

$$\int_{A\times B} f\, d\lambda \geq \int_B q^\circ\, d\mu'' - \int_A p^\circ\, d\mu'$$

$$= \int_{A\times B} f\, d\lambda^\circ \geq \int_B q\, d\mu'' - \int_A p\, d\mu'. \qquad (18)$$

(7.3;18)

fl left

All integrals in (18), except possibly the leftmost, are finite, by (4). Now, when the objective functions are well-defined as definite integrals, and finite for at least one of the two solutions being compared, standard ordering reduces to ordinary size ordering of definite integrals. Recalling that we are maximizing in the dual and minimizing in the primal, it follows at once from (18) that $(\underline{p}^{o}, \underline{q}^{o})$ and λ^{o} are best for their respective problems. ||| D

All ~~of~~ these results are direct generalizations of duality relations that hold for the ordinary transportation problem. ~~Note that~~ the finiteness condition (4) is essential to the preceding demonstrations. What happens if it is relaxed? It turns out that we can still deduce a weakened form of the conclusion of the preceding theorem, with "unsurpassed" in place of "best". Specifically we have the following result, which has an application in the theory of market regions ~~(q).~~ Section (9.5).

5a ¶ **Theorem:** Let $(\underline{p},\underline{q})$ be a measure potential for λ^{o} in the *wide sense* (i.e. without (4)), but with at least one of the two definite integrals in (4) well-defined and finite. Then λ^{o} is unsurpassed for the transportation problem (in any variant; "unsurpassed" refers to reverse standard ordering of pseudo-measures).

D

Proof: We argue by contradiction. Suppose the conclusion were false; then there is another feasible measure λ surpassing λ^o. That is,

$$\int f\, d\lambda < \int f\, d\lambda^o \qquad \text{(7.3.19)} \ \cancel{(19)}$$

(standard order; remember that we are minimizing).

Define the functions p_1, q_1 on $A \times B$ ~~as above~~ by the rules $p_1(a,b) = p(a)$, $q_1(a,b) = q(b)$. From (3) we obtain

$$\int (q_1 - p_1)\, d\lambda \leq \int f\, d\lambda \qquad \text{(7.3.20)} \ \cancel{(20)}$$

("$\leq$" refers to narrow order). Condition (14) holds for λ^o, so that inequality (3) is actually an equality λ^o-almost everywhere. From this we get the pseudomeasure equality

$$\int (q_1 - p_1)\, d\lambda^o = \int f\, d\lambda^o. \qquad \text{(7.3.21)} \ \cancel{(21)}$$

Let ψ be the pseudomeasure (λ, λ^o). Then

$$\int (q_1 - p_1)\, d\psi \leq \int f\, d\psi < 0. \qquad \text{(7.3.22)} \ \cancel{(22)}$$

(The left inequality in (22) arises on subtracting (21) from (20); the right inequality in (22) is the same as (19)). Since standard order is an extension of narrow order, (22) implies that

$$\int q_1\, d\psi < \int p_1\, d\psi. \qquad \text{(7.3.23)} \ \cancel{(23)}$$

The rest of this proof is devoted to showing that (23) is false. This contradiction will complete the proof.

superscripts +, −, ', " all on one level

One should keep in mind ~~in~~ during what follows that the upper variation ψ^+ is just an ordinary measure on $\underline{A} \times \underline{B}$; hence it has left and right marginals, which we denote by $\psi^{+\prime}$, $\psi^{+\prime\prime}$, respectively. Similarly, the lower variation ψ^- has marginals $\psi^{-\prime}$, $\psi^{-\prime\prime}$.

Consider the following indefinite integral with its Jordan form:

$$\int p_1\, d\psi = \left[\int p_1^+ d\psi^+ + \int p_1^- d\psi^-, \int p_1^- d\psi^+ + \int p_1^+ d\psi^-\right].$$

We ~~will~~ show that, for the following definite integrals,

$$\int_{\underline{A}\times\underline{B}} p_1^+ d\psi^+ = \int_{\underline{A}} p^+ d(\psi^{+\prime}) \leq \int_{\underline{A}} p^+ d(\psi^{-\prime}) = \int_{\underline{A}\times\underline{B}} p_1^+ d\psi^-,$$

~~(24)~~ (7.3.24)

and

$$\int_{\underline{A}\times\underline{B}} p_1^- d\psi^- = \int_{\underline{A}} p^- d(\psi^{-\prime}) = \int_{\underline{A}} p^- d(\psi^{+\prime}) = \int_{\underline{A}\times\underline{B}} p_1^- d\psi^+ .$$

~~(25)~~ (7.3.25)

The outer equalities in (24) and (25) all follow from the induced integrals theorem. For the middle relations in (24) and (25) consider two cases.

case ~~(i)~~ 1: Variant I or III. ~~here~~ Here

$$\lambda' = \lambda^{\circ\prime} = \mu'.$$

~~(26)~~ (7.3.26)

By the equivalence theorem for pseudomeasures we have

$$\psi^+ + \lambda^\circ = \psi^- + \lambda.$$

~~(27)~~ (7.3.27)

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μ left

Since μ' ~~being~~ is ~~sigma~~ σ-finite, there is a countable measurable partition $\mathcal{G}$ of $\underline{A}$ such that $\mu'(\underline{G}) < \infty$, all $\underline{G} \in \mathcal{G}$. For any such $\underline{G}$ and any $\underline{E} \in \Sigma'$ we have

$$\psi^{+\prime}(\underline{E} \cap \underline{G}) + \lambda^{\circ\prime}(\underline{E} \cap \underline{G}) = \psi^{-\prime}(\underline{E} \cap \underline{G}) + \lambda'(\underline{E} \cap \underline{G}),$$

on taking left marginals in (27). By (26) the λ', $\lambda^{\circ\prime}$ terms drop out. Adding over $\underline{G} \in \mathcal{G}$, we obtain

$$\psi^{+\prime}(\underline{E}) = \psi^{-\prime}(\underline{E}). \qquad (7.3.28)\ ~~(28)~~$$

Thus the left marginals of ψ^+, ψ^- are equal, and the middle relations of (24) and (25) are established with equality.

go to 18

case 2: Variant II or IV; here $\underline{p} \geq 0$, so $\underline{p}^- = 0$ and (25) is trivial. Also, since (15) holds for λ°, we have

$$\lambda^{\circ\prime}(\underline{E}) = \mu'(\underline{E}) \geq \lambda'(\underline{E})$$

for any measurable $\underline{E} \subseteq \{\underline{a} | \underline{p}(\underline{a}) > 0\}$. ~~The~~ An argument similar to (26)–(28) ~~-(28)~~ now yields

$$\psi^{+\prime}(\underline{E}) \leq \psi^{-\prime}(\underline{E})$$

for all such $\underline{E}$. This establishes the inequality in (24).

18 (24) and (25) now imply that the following statement is false:

$$\int p_1\, d\psi > 0. \qquad (7.3.29)\ ~~(29)~~$$

For (29) is true iff the sum of the two left~~-hand~~ integrals ~~in (24) and (25)~~ exceeds the sum of the two right~~-hand~~ integrals in (24) and (25).

Furthermore, we have:

$$\text{If } \int_{\underline{A}} p \, d\mu' \text{ is finite, then } \int p_1 \, d\psi \leq 0. \qquad \text{(7.3.30)} \quad \cancel{(30)}$$

To see this, note that $\psi^- \leq \lambda^o$ (minimizing property of Jordan form); hence $\psi^{-\prime} \leq \lambda^{o\prime}$, implying

$$\int_{\underline{A}} p^{\pm} \, d(\psi^{-\prime}) \leq \int_{\underline{A}} p^{\pm} \, d(\lambda^{o\prime}) \leq \int_{\underline{A}} p^{\pm} \, d\mu' < \infty.$$

This result shows that all the integrals in (24) and (25) are finite; (24) and (25) then yield the conclusion of (30).

We now run through a similar argument with $\underline{q}$ in place of $\underline{p}$. Only the high points will be mentioned. We ~~will~~ show that

$$\int_{\underline{A}\times\underline{B}} q_1^+ d\psi^+ = \int_{\underline{B}} q^+ \, d(\psi^{+\prime\prime}) \geq \int_{\underline{B}} q^+ d(\psi^{-\prime\prime}) = \int_{\underline{A}\times\underline{B}} q_1^+ d\psi^-, \qquad \text{(7.3.31)} \quad \cancel{(31)}$$

~~and~~

$$\int_{\underline{A}\times\underline{B}} q_1^- d\psi^- = \int_{\underline{B}} q^- d(\psi^{-\prime\prime}) = \int_{\underline{B}} q^- d(\psi^{+\prime\prime}) = \int_{\underline{A}\times\underline{B}} q_1^- d\psi^+. \qquad \text{(7.3.32)} \quad (32)$$

The outer equalities in (31) and (32) again follow from the induced integrals theorem. For the middle relations there are again two cases to consider.

case (i): Variant I or II; here we have

$$\lambda'' = \lambda^{o\prime\prime} = \mu''. \qquad \text{(7.3.33)} \quad (33)$$

An argument similar to (26) through (28) shows that

$$\psi^{+\prime\prime} = \psi^{-\prime\prime},$$

which establishes the middle relations of (31) and (32) with equality.

case (ii): Variant III or IV; here $q \geq 0$, so $q^- = 0$ and (32) is trivial. Also, since (16) holds for λ^o, we have

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$$\lambda^{o}{}''(F) = \mu''(F) \leq \lambda''(F)$$

for any measurable $F \subseteq \{b \mid q(b) > 0\}$. There is one subtlety at this point, since we cannot assume that λ'' or $\lambda^{o}{}''$ is sigma-finite, but the argument still goes through as follows. Let $\mathcal{G}$ be a countable measurable partition of B such that $\mu''(G) < \infty$, all $G \in \mathcal{G}$. Then, for any such F and G, we have (noting that $\lambda^{o}{}''(F \cap G)$ is finite),

$$\begin{aligned} \psi^{+}{}''(F \cap G) &= \psi^{-}{}''(F \cap G) + \lambda''(F \cap G) - \lambda^{o}{}''(F \cap G) \\ &\geq \psi^{-}{}''(F \cap G). \end{aligned}$$

Adding over $G \in \mathcal{G}$, we obtain

$$\psi^{+}{}''(F) \geq \psi^{-}{}''(F)$$

for all measurable $F \subseteq \{b \mid q(b) > 0\}$, which establishes the inequality in (31).

Parallel to the argument above, (31) and (32) now imply that the following statement is false:

(7.3.34)

$$\int q_1 \, d\psi < 0, \qquad (34)$$

and, furthermore,

If $\int_B q \, d\mu''$ is finite, then $\int q_1 \, d\psi \geq 0$. ~~(35)~~ (7.3.35)

Condition (35) follows from the observations:

$$\int_B q^{\pm} \, d(\psi^{-''}) \leq \int_B q^{\pm} \, d(\lambda^{\circ''}) = \int_B q^{\pm} \, d\mu'' < \infty. \qquad \text{~~(36)~~ (7.3.36)}$$

(The equality in (36) follows from (33) in variant I or II; it follows from $q \geq 0$ and (16) in variant III or IV). Then (36), (31), and (32) ~~then~~ imply that

$$\int_{A \times B} q_1^{+} \, d\psi^{-} + \int_{A \times B} q_1^{-} \, d\psi^{+} < \infty ,$$

and this fact, together with (31) and (32), yields the conclusion in (35).

Finally, at least one of the premises in (30) or (35) is true by assumption; hence at least one of the conclusions in (30) or (35) is true. Thus, combining things with (23), either

$$\int q_1 \, d\psi < \int p_1 \, d\psi \leq 0,$$

which contradicts the falsity of (34), or

$$0 \leq \int q_1 \, d\psi < \int p_1 \, d\psi,$$

which contradicts the falsity of (29). This contradiction completes the proof. □

~~We conclude the duality section at this point. There are a number of unanswered questions, even for the case when (4) holds. For example, is (17) a necessary condition for solutions~~

~~to be optimal, or can a positive gap exist between optimal primal and dual values? We shall tackle the question of whether an optimal λ° has a measure potential in a later section.~~

(A)

7.4. The Transportation Problem: Existence of Optimal Solutions

~~This book is now fairly well along, and~~ so far in this book ~~up to this point~~ topological concepts have scarcely been used. This has been a matter of deliberate policy, to underline the fact that — contrary to popular belief ~~table-talk~~ — measure theory is far more significant than topology as a groundwork for social science.

But for the rest of this section, topology is essential. Indeed, we know of no general method for proving existence of optimal solutions to the measure-theoretic transportation problem without using topological concepts. Nor do we know of any topology-free method for constructing measure potentials from optimal solutions.

16 We ~~shall~~ do not go deeply into the subject, but simply list those concepts and theorems ~~which~~ that are actually used in the sequel.[6]

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18 Given a fixed set $\underline{A}$, let T be a collection of subsets of $\underline{A}$. Then T is a topology over $\underline{A}$ iff (i) $\underline{A} \in T$, and the empty set $\emptyset \in T$; (ii) if $G_1, G_2 \in T$, then $G_1 \cap G_2 \in T$; (iii) if $\mathcal{G} \subseteq T$, then $\cup\mathcal{G} \in T$. (In words: T is closed under arbitrary unions and finite intersections, and owns $\underline{A}$ and $\emptyset$). The pair $(\underline{A},T)$ is a topological space and the members of T are called open sets.

Set F is said to be closed iff $A \setminus F$ is open.

Let $\mathcal{G}$ be any class of subsets of A. From this, construct $\mathcal{G}'$, the class of intersections of finite subclasses of $\mathcal{G}$, and then construct $\mathcal{G}''$, the class of unions of arbitrary subclasses of $\mathcal{G}'$, together with A and $\emptyset$. One can show that $\mathcal{G}''$ is a topology, the topology generated by $\mathcal{G}$. $\mathcal{G}$ is called a subbase, and $\mathcal{G}'$ a base, for this topology.[7]

script T

Let $(A, \overset{T}{\emptyset})$ be a topological space and let $E \subseteq A$. The relative topology on E is the class of all sets of the form $E \cap G$, G ranging over the open sets of A. This collection of sets makes E a topological space in its own right. This is the construction implicitly referred to below when we speak of, ~~e.g.,~~ the "usual topology of the rationals", or "topological completeness of a closed subset of A."

Here ~~are some~~ is an example~~s~~. Let A be the real line, and let $\mathcal{G}$ be the class of all open intervals. The topology generated by $\mathcal{G}$ is the usual topology for the real line, the one implicitly used in ordinary discussions of continuity and convergence of real numbers.

This example has a far-reaching generalization. Recall that (A, d) is a metric space iff the function $d: A \times A \to$ reals satisfies: $d(a, a) = 0$, $d(a_1, a_2) > 0$ if $a_1 \neq a_2$, $d(a_1, a_2) = d(a_2, a_1)$, $d(a_1, a_2) + d(a_2, a_3) \geq d(a_1, a_3)$. Now let $\mathcal{G}$ be the class of all open discs — ~~that is~~ i.e., all sets of the form $\{a \mid d(a_0, a) < x\}$, for $a_0 \in A$, x real and positive. The topology generated by the metric d is the one generated by this $\mathcal{G}$.

The topology generated by the Euclidean metric in n-space ($n = 1, 2, \ldots$) is the usual topology for this space, and will be assumed if no contrary assumptions are made.

Given a topological space, $(A, \mathcal{T})$, ~~one~~ can ask: Does ~~there~~ exist a metric d on A that ~~which~~ generates $\mathcal{T}$ as described above? $\mathcal{T}$ is said to be <u>metrizable</u> iff this is the case. script T

Sc 1 12 We need the concept of <u>completeness</u> of a metric space. A sequence $a_1, a_2, \ldots$ from (A, d) has the <u>Cauchy</u> property iff, for all $\varepsilon > 0$, there is an integer N such that, for all integers $m, n > N$, $d(a_m, a_n) < \varepsilon$: roughly speaking, the points get indefinitely close to each other. Space (A, d) is <u>complete</u> iff any such Cauchy sequence converges to a point $a_o \in A$. i.e. ~~That is~~, for all $\varepsilon > 0$, ~~there is~~ an N exists such that, for all $n > N$, $d(a_o, a_n) < \varepsilon$. For example, the real numbers are complete and the rationals are not, under the usual topology. A space $(A, \mathcal{T})$ that ~~which~~ is not only metrizable, but generated by a <u>complete</u> metric is said to be <u>topologically complete</u>. The real line, and n-space in general, are topologically complete under the usual topologies, and the same is true for any closed or open subsets of these spaces. ~~(in the relative topology).~~ script T

Space $(A, \mathcal{T})$ is <u>separable</u> iff there is a countable subset A' such that any non-empty open set meets A': $G \in \mathcal{T}$ and $G \neq \emptyset$ implies $G \cap A' \neq \emptyset$. The n-space is separable for any n (e.g., A' may be chosen to be the set of n-tuples with rational coordinates). Indeed, <u>any</u> subset of n-space is separable.

~~(under the topology generated by the Euclidean metric restricted to the subset).~~

A subset K of a metrizable topological space is said to be compact iff every sequence from it contains a subsequence converging to a point of K. Any closed bounded subset of n-space, for example, is compact.

We need a few continuity concepts, confining our attention to real-valued functions. Let (A, T) be a topological space, and f a real-valued function with domain A. f is lower semi-continuous iff every set of the form $\{a|f(a) > x\}$, x real, is open.[8] f is upper semi-continuous iff $-f$ is lower semi-continuous, or, equivalently, iff all sets of the form $\{a|f(a) < x\}$ are open. Finally, f is continuous iff it is both lower and upper semi-continuous.

Let (A, T') and (B, T'') be two topological spaces. On the cartesian product $A \times B$ one defines a product topology, written $T' \times T''$, essentially in the same way one defines product sigma-fields. Specifically, rectangles $E \times F$ are called open iff $E \in T'$ and $F \in T''$; $T' \times T''$ is then defined to be the topology generated by the open rectangles. One can show, in fact, that $G \in T' \times T''$ iff G is the union of some subclass of open rectangles.

The discussion to this point has been "purely" topological. We also need some concepts that depend both on topological and measure-theoretic ideas. The Borel field of topological space (A, T) is the sigma-field generated by T.[9] Its members are called **Borel sets**.

Let (A, T) be a metrizable topological space, let Σ be its Borel field, and let μ be a bounded measure on Σ. μ is said to be **tight** iff, for every positive number ε, there is a compact set K such that $\mu(A \setminus K) < \varepsilon$. Let $\mathfrak{M}$ be a collection of measures on Σ. $\mathfrak{M}$ is **uniformly tight** iff (i) there is a real number N such that $\mu(A) \leq N$ for all $\mu \in \mathfrak{M}$, and (ii) for every positive number ε there is a compact K such that $\mu(A \setminus K) < \varepsilon$ for all $\mu \in \mathfrak{M}$. (~~Note that~~ the **same** K must serve for all μ).

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$\mu(A \setminus K)$

With (A, T) and Σ as above, let μ_o be a bounded measure on Σ, and $\mu_1, \mu_2, \ldots$ a sequence of such measures. This sequence is said to **converge weakly** to μ_o iff, for every $g: A \to$ reals ~~which~~ that is bounded and continuous we have

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$$\lim_{n\to\infty} \int_A g \, d\mu_n = \int_A g \, d\mu_o \qquad \text{(7.4.1)} \ \ \cancel{(1)}$$

(in the ordinary sense of limit of a sequence of real numbers).

~~The set of bounded measures $\mathfrak{M}$ is said to be **weakly relatively compact** iff every sequence of measures in $\mathfrak{M}$ contains a subsequence which converges weakly to some bounded measure (not necessarily belonging to $\mathfrak{M}$).~~

18 With these definitions out of the way, we are ready to proceed. The following is an omnibus theorem, covering the alternative variants of the transportation problem under alternative assumptions. ~~Let us~~ We first briefly contemplate the

practical import of some of these assumptions. Any subset of n-space is separable (in the relative topology), and any subset ~~which~~ that can be expressed as a countable intersection of open sets (a so-called G_δ-set) is topologically complete (Aleksandrov's theorem; in ~~(in~~ particular, closed or open subsets of n-space are G_δ). Hence these conditions constitute no real restriction when imposed on (physical) Space or Time, or other spaces built up from these in a simple manner. It is unclear what restrictions they imply when imposed on topologies on more complex spaces, such as Resources, ~~or~~ Histories, or Activities. The boundedness of f, μ', and μ" means that these results will often not apply to problems involving infinite ~~Space~~ or ~~Time~~ horizons, but they constitute no real restriction when applied to "practical" problems in the narrow sense of the term.

~~Note also that~~ these boundedness assumptions also imply that the objective function is well-defined and finite as a definite integral for any feasible flow λ. Hence standard order reduces to the ordinary comparison of definite integrals. We may ~~also~~ therefore drop the distinction between "best" and "unsurpassed" solutions, and simply speak of optimal solutions.

Theorem: (omnibus existence theorem). Let (A, Σ', μ'), (B, Σ", μ") be the source and sink spaces for the transportation problem, B ≠ Ø, with cost function f: A × B → reals, where

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(i) μ', μ", and f are all bounded, and μ'(A) ≥ μ"(B); ~~and~~

(ii) ~~there~~ exist topologies T', T'' on A, B, with Σ', Σ'' as their Borel fields, respectively; T', T'' are both topologically complete and separable; and f is lower semi-continuous with respect to $T' \times T''$; ~~and~~

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(iii) one of the following extra assumptions is made:

(a) no extra assumption, or

(b) $f \geq 0$, or

(c) (B, T'') is compact, or

(d) $\mu'(A) = \mu''(B)$.

Then we have

110

equal-size boxes

Use cap rom E in place of √ if √ is not on disc

181	24 II	31 IV	31 III	24 I
(a)	✓	×	×	×
(b)	✓	✓	×	×
(c)	✓	✓	✓	×
(d)	✓	✓	✓	✓

× = times

Set 1/2 pt BLR Proof flag for rule & add vertical rules during pg me

where ✓ indicates that ~~there~~ always exists an optimal solution to the transportation problem for the given combination of variant I, II, III, or IV, and assumption (a), (b), (c), or (d), and × indicates that ~~there~~ sometimes does not exist such a solution.

Proof: There are sixteen things to prove, and of these twelve can be disposed of rapidly, either directly or as a consequence of the other four. First, we discuss the counterexamples for the "×" (times) entries.

\# Cases (Ia), (Ib), (Ic): Let A and B be singleton sets, with $\mu(A) > \mu(B)$. Then variant I does not have even a feasible, let alone an optimal, solution.

18

\# Cases (IIIa), (IIIb): Let A be singleton, with $\mu'(A) = 1$; let B be the integers $\{1, 2, \ldots\}$, with $T'' = \Sigma'' =$ all subsets of B, and μ'' identically zero; finally, let unit transport cost to point $n \in B$ be $1/n$, ($n = 1, 2, \ldots$).

18

Since B is countable, T'' is separable; it is also topologically complete, since it is generated by the complete metric d given by $d(m, n) = 1$ if $m \neq n$. Thus the stated problem satisfies all premises of the theorem, including (a) and (b) of (iii).

There is no optimal solution, ~~For, on the one hand,~~ cost can be reduced below any $\varepsilon > 0$ by shipping one unit to a sufficiently large $n \in B$, ~~On the~~ but other hand, zero cost cannot be attained, since $f > 0$, and $\lambda(A \times B) = \mu'(A) > 0$.

\# Case (IVa): Let everything be as in the counterexample just given, except that unit transport cost to point $n \in B$ is $(1-n)/n$.

18

Cost can be reduced below any real number >-1 by shipping one unit to sufficiently large $n \in B$, ~~On the~~ but other hand, cost of -1 cannot be attained. This is clear if $\lambda = 0$; while if $\lambda(A \times B) > 0$, then, since $f > -1$, we have

$$\int_{\underline{A}\times\underline{B}} f\, d\lambda > \int_{\underline{A}\times\underline{B}} -1\, d\lambda = -\lambda(\underline{A}\times\underline{B}) \geq -\mu'(\underline{A}) = -1.$$

Hence again there is no optimal solution.

Cases (IIb) (IIc) (IId): This finishes the "x"s, and we start on the "$\underline{E}'$"s. First ~~of all~~, existence in cases (IIb), (IIc), and (IId) obviously follows from existence in case (IIa).

Cases (Id), (IIId), (IVd): Existence in these cases follows from existence in case (IId). To show this, we prove that any feasible flow measure λ must satisfy the transportation problem constraints with equality in all four variants. For suppose there were an $\underline{E} \in \Sigma'$ such that $\mu'(\underline{E}) > \lambda'(\underline{E})$. Adding the inequality $\mu'(\underline{A}\backslash\underline{E}) \geq \lambda'(\underline{A}\backslash\underline{E})$, we obtain $\mu'(\underline{A}) > \lambda'(\underline{A})$. (The strict inequality carries over because all measures involved are finite). But $\lambda'(\underline{A}) = \lambda(\underline{A}\times\underline{B}) \geq \mu''(\underline{B})$, so $\mu'(\underline{A}) > \mu''(\underline{B})$, contradicting premise (d). Thus we must have $\lambda' = \mu'$; a similar argument establishes $\lambda'' = \mu''$. Hence all four variants have the same set of feasible solutions. If an optimal solution exists for any of them, therefore, it must exist for all, under premise (d).

This leaves the four cases: (IIa), (IIIc), (IVc), and (IVb).

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18 Case (IVb): We show that the existence of an optimal solution in case IIb implies its existence in IVb. Let λ be feasible for variant IV, so that $\lambda'' \geq \mu''$. Thus μ'' is absolutely continuous with respect to λ''. Since all measures are also finite, we may invoke the Radon-Nikodym theorem and infer the existence of a function $g:B \rightarrow$ reals satisfying

$$\mu'' = \int g \, d\lambda''. \qquad (7.4.2)$$

Then g must take on values in the closed interval $[0,1]$, except possibly on a Σ''-set of λ''-measure zero. Altering it to zero on this set, (which does not invalidate (2)), we thus have $0 \leq g(b) \leq 1$, for all $b \in B$. Now define $h:A \times B \rightarrow [0,1]$ by: $h(a,b) = g(b)$, ~~and~~ then define λ_1 by

$$\lambda_1 = \int h \, d\lambda. \qquad (7.4.3)$$

This is an indefinite integral over $A \times B$, and λ_1 is therefore another flow measure on $(A \times B, \Sigma' \times \Sigma'')$. We now show it is feasible for variant II. First ~~of all~~, $\lambda_1 \leq \lambda$, since $h \leq 1$; hence

$$\lambda_1' \leq \lambda' \leq \mu'. \qquad (7.4.4)$$

Also

$$\lambda_1'' = \int g \, d\lambda'', \qquad (7.4.5)$$

from (3) by the induced integrals theorem. Relations (4), (5), and (2) show that λ_1 is feasible for variant II. Also, since $\lambda_1 \leq \lambda$

and $f \geq 0$, we obtain

$$\int_{A\times B} f\,d\lambda_1 \leq \int_{A\times B} f\,d\lambda. \qquad (7.4.6)\ \cancel{(6)}$$

Thus, assuming $f \geq 0$, we have shown that for any flow λ feasible in variant IV, ~~there~~ exists a flow λ_1 ~~which~~ that meets the more stringent conditions of variant II, and whose transportation cost is no higher than that of λ, by (6). It follows that any solution optimal for variant II will also be optimal for variant IV, under premise (b).

Cases (IIa), (IIIc), (IVc): This is the last, and most difficult, part.[10] The proof goes through several stages. The first ~~stage~~ is to show that the set of feasible solutions in these cases is uniformly tight.

~~First of all,~~ from the fact that T' and T'' are both separable and metrizable, it may be shown that $\Sigma' \times \Sigma''$ is the Borel field of $T' \times T''$. Also it is known that any bounded measure on the Borel field of a topologically complete and separable topological space is tight (Ulam~~; see Billingsley, pages 5-6~~ theorem). Hence, for any $\varepsilon > 0$ there are compact sets K', K'' — contained in A, B, respectively — such that

$$\mu'(A\setminus K') < \varepsilon/2, \quad \mu''(B\setminus K'') < \varepsilon/2. \qquad (7.4.7)\ \cancel{(7)}$$

Now let λ be a feasible flow measure. We always have $\lambda'(A\setminus K') \leq \mu'(A\setminus K')$. In variant II we also have

$$\lambda''(B\backslash K'') = \mu''(B\backslash K''). \qquad (7.4.8)\ \ \cancel{(8)}$$

Furthermore, if premise (c) holds, B itself is compact and we may choose K" = B to satisfy (7) and (8). Hence in all the three cases (IIa), (IIIc) and (IVc) K' and K" may be chosen so that

$$\lambda'(A\backslash K') + \lambda''(B\backslash K'') < \varepsilon \qquad (7.4.9)\ \ \cancel{(9)}$$

for all feasible λ. Next, consider the number $\lambda\left[(A\times B)\backslash(K'\times K'')\right]$. This does not exceed the left side of (9), since

$$(A \times B)\backslash(K' \times K'') \subseteq [(A\backslash K') \times B] \cup [A \times (B\backslash K'')]$$

Also K' × K" is itself compact in the product topology $T' \times T''$ (Tihonov's theorem). Finally, $\lambda(A \times B) \leq \mu'(A) < \infty$ for all feasible λ. This shows that the set of feasible solutions is indeed uniformly tight in all three cases.

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18 We know there is at least one feasible solution (section 7.2). Hence there is a sequence of them, $\lambda_1, \lambda_2, \ldots$, with the property

$$\lim_{n} \int_{A\times B} f_1\, d\lambda_n = V^{\circ} \qquad (7.4.10)\ \ \cancel{(10)}$$

where V° is the infimum of the attainable values of the objective function.

Since T' and T'' are both metrizable, the same may be shown to be true for the product topology. We now invoke the basic theorem of Prohorov-Varadarajan. ~~(Billingsley, page 32):~~ If $\lambda_1, \lambda_2, \ldots$ is a sequence from a uniformly tight set of measures on the Borel field of a metrizable topological space, then there is a subsequence ~~which~~ that converges weakly to some measure λ^o (not necessarily a member of the set).

Let λ^o be this weak limit of a subsequence of the $\lambda_1, \lambda_2, \ldots$ ~~which~~ that satisfies (10). We ~~shall~~ prove ~~that~~ this λ^o is in fact optimal for the transportation problem by showing that it is feasible and that

$$\int_{A \times B} f \, d\lambda^o \leq V^o. \tag{7.4.11}$$

~~(11)~~

We ~~First for~~ prove (11) first. For convenience we retain the notation $\lambda_1, \lambda_2, \ldots$ for the convergent subsequence of the original sequence. Invoking a theorem of A. D. Aleksandrov, it follows that

$$\lambda^o(G) \leq \liminf_n \lambda_n(G) \tag{7.4.12}$$

~~(12)~~

for all open sets $G \subseteq A \times B$. Let us now temporarily make the additional assumption that $f \geq 0$. We then have

$$\begin{aligned} \int_{A \times B} f \, d\lambda^o &= \int_0^\infty \lambda^o\{(a,b) \mid f(a,b) > t\}\,dt \\ &\leq \int_0^\infty \liminf_n \lambda_n\{(a,b) \mid f(a,b) > t\}\,dt \\ &\leq \liminf_n \int_0^\infty \lambda_n\{(a,b) \mid f(a,b) > t\}\,dt \\ &= \liminf_n \int_{A \times B} f \, d\lambda_n = V^o. \end{aligned} \tag{7.4.13}$$

~~(13)~~

0 = zero where used

(The last equality in (13) comes from (10)). The other two equalities invoke the Young integral, which is an ordinary Riemann integral ~~see page~~. The first inequality results from Aleksandrov's theorem, (12), and the fact that – since f is lower semi-continuous – the set $\{(a,b) \mid f(a,b) > t\}$ is open. The second inequality is from Fatou's lemma).

Now drop the assumption that $f \geq 0$. Since f is bounded below, ~~there is~~ a real number x exists such that $f + x \geq 0$. The argument of (13) now yields

$$\int_{A\times B} (f+x)\, d\lambda^{\circ} \leq \liminf_{n} \int_{A\times B} (f+x)\, d\lambda_n. \qquad (7.4.14)\ \cancel{(14)}$$

But weak convergence implies that $\lim \lambda_n(A \times B) = \lambda^{\circ}(A \times B)$. (Substitute the constant function $g = 1$ in ~~definition~~ (1)). It follows that the x's drop out of (14). Thus we obtain (11).

The final step is to prove that λ° is feasible. ~~First of all,~~ Since $\lambda_1, \lambda_2, \ldots$ converges weakly to λ°, it follows that the marginals converge weakly to the respective marginals of λ° (by the Mann-Wald theorem ~~; Billingsley, pages 30-31~~):

$$\lim \lambda_n' = \lambda^{\circ\prime}, \quad \lim \lambda_n'' = \lambda^{\circ\prime\prime},$$

where "lim" stands for weak convergence. ~~Now,~~ For all n, the marginals satisfy the feasibility constraints for the transportation problem:

$$\lambda_n' \ (=, \leq)\ \mu', \quad \lambda_n'' \ (=, \geq)\ \mu'', \qquad (7.4.15)\ \cancel{(15)}$$

the particular signs depending on the variant in question. We must show that the marginals of λ^{o} satisfy the same constraints.

g22-5 The following result will be used: Let μ, ν be two bounded measures on the Borel field of a metrizable topological space, C. If

$$\int_C g\, d\mu \geq \int_C g\, d\nu \qquad \text{(7.4.16)} \quad \cancel{(16)}$$

for all bounded non-negative continuous functions g, then $\mu \geq \nu$.[14] It follows that, if $\mu_1, \mu_2, \ldots$, and $\nu_1, \nu_2, \ldots$ are two sequences of bounded measures on this Borel field, converging weakly to bounded measures μ, ν, respectively, and $\mu_n \geq \nu_n$ for all n, then $\mu \geq \nu$. For,

$$\int_C g\, d\mu = \lim_n \int_C g\, d\mu_n \geq \lim_n \int_C g\, d\nu_n = \int_C g\, d\nu,$$

if g is bounded non-negative continuous; thus (16) holds, yielding $\mu \geq \nu$.

Now consider the two sequences: $\mu', \mu', \ldots$, and $\lambda_1', \lambda_2', \ldots$. The first is a constant sequence, converging weakly to μ'; the second converges weakly to $\lambda^{o\prime}$. Since $\mu' \geq \lambda_n'$, all n, we obtain $\mu' \geq \lambda^{o\prime\prime}$ by the above result. Similar arguments show that the relations (15) get reproduced with $\lambda^{o\prime}$, $\lambda^{o\prime\prime}$ in place of λ_n', λ_n'', respectively. Hence λ^{o} is indeed feasible.

Since it also attains the infimum of the objective function, by (11), it is optimal. ~~The proof is complete.~~ ∎

These results can be generalized in a number of ways. First~~of all~~, the condition that the unit transport cost function f be bounded can be relaxed. The fact that f is bounded above was used only to guarantee that $\int_{A\times B} f\, d\lambda^\circ$ is finite. Hence we need assume merely that f is bounded below and that $\int_{A\times B} f\, d\lambda$ is finite for at least one feasible λ.

The condition that f be bounded below can in turn be weakened to the following: ~~There~~ exists a measurable function $h: A \to$ reals such that $h(a) \leq f(a,b)$, all $a \in A$, $b \in B$, and $\int_A h\, d\mu'$ is finite. We omit the proof of this statement.

~~Next~~ Second, the condition that $(A, \mathcal{T}')$ and $(B, \mathcal{T}'')$ be topologically complete and separable can be weakened to their being merely Borel subsets of such spaces. For, ~~first of all~~, such subsets still remain separable metrizable, and, ~~secondly~~, any bounded measure on such a space still remains tight. VS ~~(Parthasarathy, pages 29-30)~~. The proof then proceeds exactly as above. Just about any subsets of n-space, with the usual topology, ~~which~~ that arise in practice would fulfill this condition, ~~for example~~.

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Finally, the simple constraints of the transportation problem can be complicated considerably without invalidating the preceding proof. Suppose, for example, that shipments are not allowed between certain source-sink pairs (say because a road does not exist, or because the resources at the sources are unsuitable for the activities at the sinks). More generally, there may be upper limits on the flows between certain pairs (e.g.) (due, ~~for example,~~ to limited road capacity). Or, there may be lower limits. The following result gives the generalization.

Theorem: Let a transportation problem satisfy all the premises of the preceding theorem for one of the checked cases (IIa, etc.). In addition to the usual constraints, any feasible flow λ is required to satisfy the following:

$$\lambda(G_i) \leq x_i, \qquad \text{(7.4.17)} \ \sout{(17)}$$

~~and~~ all $i \in I$, and

$$\lambda(F_j) \geq y_j, \qquad \text{(7.4.18)} \ \sout{(18)}$$

all $j \in J$. (Here I and J are arbitrary sets indexing the constraints, the x_i and y_j are given real numbers, the G_i and F_j are given open and closed subsets, respectively, of $A \times B$.)

Then, provided at least one flow λ exists satisfying the transportation problem constraints augmented by (17) and (18), ~~there~~ exists an optimal solution for this system of constraints.

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Proof: The proof proceeds exactly as in the omnibus theorem, with these additional comments. The set of feasible solutions here is a subset of the original feasible set, hence it inherits the uniform tightness of the original. As above, there is a sequence of feasible flows $\lambda_1, \lambda_2, \ldots$ converging weakly to a bounded flow λ°, and the objective function $\int_{A\times B} f_1 \, d\lambda$ attains its infimum at λ°.

It remains to prove that λ° is feasible. It satisfies the original transportation problem constraints, by the proof above; and we must show it satisfies (17) and (18). But

$$\liminf_{n} \lambda_n(G_i) \leq x_i$$

for all $i \in I$, since each of $\lambda_1, \lambda_2, \ldots$ satisfies (17). This, with Aleksandrov's theorem, (12), proves that λ° satisfies (17). Similarly, we have

$$\lambda^{\circ}(F_j) \geq \limsup_{n} \lambda_n(F_j) \geq y_j \qquad \text{(7.4.19)} \quad \cancel{(19)}$$

for all $j \in J$. The left inequality in (19) is a corollary of (12), which holds for any closed set F_j; the right inequality arises from the fact that all $\lambda_1, \lambda_2, \ldots$ satisfy (18). Relation (19) implies that λ° itself satisfies (18). Hence it is feasible. |||

7.5. The Transportation Problem: Existence of Potentials

We have seen that, if a pair of functions $p:A \to$ reals, $q:B \to$ reals is a measure potential for flow measure λ, ~~then~~ the latter is optimal for the transportation problem. In this section we tackle the converse question: Given an optimal solution to the transportation problem, does ~~there~~ exist a pair of functions that which is a measure potential for λ?

Experience indicates that questions of this sort are hard to answer, and this one is no exception. Our procedure will be to establish the existence of functions with a slightly different property, that of being a <u>topological</u> potential for λ. We begin by setting out the various concepts needed, and investigating the relation between the two "potential" concepts.

\# 18 Let $(C, \mathcal{T})$ be a topological space. Set $E \subseteq C$ is a <u>neighborhood</u> of point $c \in C$ iff there is an open set G for which $c \in G$ and $G \subseteq E$. Now let C also be a measure space, with ~~sigma~~ σ-field Σ and measure μ. (We make no assumptions about the relation between $\mathcal{T}$ and Σ); $c \in C$ is a <u>point of support</u> of the measure μ (with respect to $\mathcal{T}$) iff every measurable neighborhood of c has positive μ-measure. The set of all points of support is called simply the <u>support</u> of μ.

Intuitively, the support of a measure is "where it's at". As examples, ~~let~~ we us take some familiar probability measures on the real line, with $\mathcal{T}$ and Σ the usual topology and Borel field. For a discrete distribution, taking positive mass on at most a finite number of points, the support is simply those points. For the Poisson distribution, it is the non-negative integers. For the normal distribution, it is the entire line. It may be shown that the support of a measure is always a <u>closed</u> set. (Hint: show that the complement is open.)

\# 18 Now suppose we are given a transportation problem defined by the source and sink spaces (A, Σ', μ'), (B, Σ'', μ''), respectively, and unit cost function $f: A \times B \to$ reals. As always, μ', μ'' are ~~sigma~~ σ-finite and f is measurable. Also assume that A and B are furnished with topologies $\mathcal{T}'$, $\mathcal{T}''$, respectively. The following definition applies to any of the variants - <u>I</u>, <u>II</u>, III, or IV — of the transportation problem. Recall that all variants have the same dual — (3.3) to (3.5) ~~(3), (4), and (5) of section 3~~ — except for sign restrictions: $p \geq 0$ in variants II and IV, $q \geq 0$ in variants III and IV.

Definition: Let flow measure λ on $(A \times B, \Sigma' \times \Sigma'')$, and the pair of functions $p: A \to$ reals, $q: B \to$ reals be feasible for the transportation problem and its dual, respectively. Pair (p,q) is a topological potential for λ iff the following ~~three~~ conditions are satisfied:

(i) If (a,b) is a point of support for λ, then

$$q(b) - p(a) = f(a,b); \qquad (7.5.1)$$

(ii) If $a \in A$ is a point of support for $(\mu' - \lambda')$, then

$$p(a) = 0; \qquad (7.5.2)$$

(iii) If $b \in B$ is a point of support for $(\lambda'' - \mu'')$, then

$$q(b) = 0. \qquad (7.5.3)$$

A few clarifying comments are in order. "Point of support" in (1) refers of course to the product space $A \times B$, and it is relative to the product topology $T' \times T''$. In (2) and (3) we are dealing with measures on (A, Σ') and (B, Σ''), relative to the topologies T' and T'', respectively.

Measure $\mu' - \lambda'$ is capacity minus outflow, and so is the unused capacity measure on the source space A. Similarly, $\lambda'' - \mu''$ is inflow minus requirement, and so is the oversupply measure on the sink space B. (Subtraction of measures is defined as in ~~chapter 3,~~ section 3.1.)

~~As we stated,~~ this definition is to apply to all four variants. Note, however, that for variants I and III, $\mu' = \lambda'$; hence $\mu' - \lambda' = 0$ has no points of support. (2) is then vacuously true and may be dropped from the conditions. Similarly, in variants I and II condition (3) is vacuously true and may be dropped. This is exactly as in the definition of measure potential.

There is, indeed, a striking parallelism between the two "potential" concepts. Conditions (1)–(3) have as much claim to generalize the complementary slackness conditions as do the corresponding conditions (3.14)–(3.16) ~~(14)-(16) of section 3~~. For the special case in which the sigma-fields Σ' and Σ'' are finite (and coincide with the respective topologies $\mathcal{T}'$ and $\mathcal{T}''$), both potential concepts reduce to the ordinary complementary slackness conditions. That ($\underline{a}$,$\underline{b}$) ~~being~~ is a point of support for λ generalizes the notion ~~that there is~~ of a positive flow from source $\underline{a}$ to sink $\underline{b}$. Complementary slackness requires that the dual relation for the pair ($\underline{a}$,$\underline{b}$) be fulfilled with equality, and this is just what (1) requires. Again, if there is unused capacity at a source, the dual variable must be zero; this is (2). Condition (3) generalizes the analogous condition for oversupplied sinks.

Topological potentials in the wide sense are defined in the same way as the corresponding wide-sense concept for measure potentials, ~~namely,~~ viz. by dropping the requirement that ($\underline{p}$,$\underline{q}$) must satisfy the finiteness condition, (3.4) ~~(4) of section 3~~. In the following discussion of the relation between these two "potential" concepts, we understand them to be either both ordinary or both wide-sense.

We are mainly interested in determining when a topological potential will also be a measure potential; for a topological potential is what we get from the theorems to come, while a measure potential is what we want. The following concept is needed.

Definition: A topological space has the strong Lindelöf property iff, for every collection of open sets G, there is a countable subcollection $G' \subseteq G$ such that $\cup G' = \cup G$.

Any subset of n-space with the topology generated by the Euclidean metric — (indeed, any separable metrizable space) — has this property, so that it includes many cases of practical interest.[~~12~~ 16] The following theorems apply to all four variants of the transportation problem.

Theorem: Given a transportation problem, and given topologies T', T'' on the source and sink spaces, A, B, respectively; if (p,q) is a topological potential for flow λ, and the product space ($A \times B$, $T' \times T''$) has the strong Lindelöf property, then (p,q) is a measure potential for λ.

Proof: We show that (1)–(3) ~~(1), (2), (3)~~ imply the corresponding conditions for measure potentiality, (3.14)–(3.16) ~~(14), (15), (16) of section 3~~, respectively. If

$$q(b) - p(a) < f(a,b) \qquad \text{(7.5.4)} \ \text{~~(4)~~}$$

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for a point (a,b), then (a,b) is not a point of support for λ, by (1). Hence (a,b) has a measurable neighborhood, $N_{(a,b)}$, of λ-measure zero. There is an open set $G_{(a,b)} \subseteq N_{(a,b)}$ with $(a,b) \in G_{(a,b)}$. Consider the collection, $\mathcal{G}$, of all these open sets, one for each point (a,b) satisfying (4). By the strong Lindelöf property, there is a countable subcollection $\mathcal{G}'$ with $\cup\mathcal{G}' = \cup\mathcal{G}$. Let $\mathcal{N}'$ be the corresponding subcollection of neighborhoods of measure zero; $\mathcal{N}'$ is also countable. Then

$$\{(a,b)\,|\,q(b) - p(a) < f(a,b)\} \subseteq (\cup\mathcal{G}) = (\cup\mathcal{G}') \subseteq (\cup\mathcal{N}').$$

Hence $\lambda\{(a,b)\,|\,q(b) - p(a) < f(a,b)\}$ does not exceed the sum of $\lambda(N_{(a,b)})$ over all members of $\mathcal{N}'$. This sum being zero, we obtain (3.14) ~~(14) of section 3~~.

Next we prove (3.15) ~~(15) of section 3. First,~~ it is easily verified that the component spaces $(A, \mathcal{T}')$ and $(B, \mathcal{T}'')$ inherit the strong Lindelöf property from $(A \times B, \mathcal{T}' \times \mathcal{T}'')$. If $p(a) > 0$ for point $a \in A$, then a is not a point of support for $\mu' - \lambda'$, by (2). Arguing as above, we find that $\{a\,|\,p(a) > 0\}$ is covered by a countable number of sets of $(\mu' - \lambda')$-measure zero. Hence $(\mu' - \lambda')\{a\,|\,p(a) > 0\} = 0$, which implies that $\mu' = \lambda'$ on subsets of $\{a\,|\,p(a) > 0\}$. This is (3.15). ~~The opposite inequality being always true, we obtain (15) of section 3.~~

In the same way we find that $(\lambda'' - \mu'')\{b\,|\,q(b) > 0\} = 0$, which implies that $\lambda'' = \mu''$ on subsets of $\{b\,|\,q(b) > 0\}$, yielding (3.16) ~~(16) of section 3~~. |||

A condition for the opposite implication is easier to find and to prove:

Theorem: Let (p,q) be a measure potential for flow λ. Let T', T'' be topologies on the source and sink spaces A, B, respectively, such that

$$\{(a,b)\,|\,q(b) - p(a) < f(a,b)\} \in T' \times T'', \qquad \text{(7.5.5)} \ \cancel{(5)}$$

and (in variants II and IV)

$$\{a\,|\,p(a) > 0\} \in T', \qquad \text{(7.5.6)} \ \cancel{(6)}$$

and (in variants III and IV)

$$\{b\,|\,q(b) > 0\} \in T''. \qquad \text{(7.5.7)} \ \cancel{(7)}$$

Then (p,q) is a topological potential for λ.

Proof: We show that (3.14)–(3.16) ~~(14), (15) and (16) of section 3~~ imply the corresponding conditions (1)–(3) ~~(1), (2), (3)~~ for topological potentiality, respectively.

Let (1) be false, so that λ has a point of support (a_o,b_o) for which $q(b_o) - p(a_o) < f(a_o,b_o)$. The set $\{(a,b)\,|\,q(b) - p(a) < f(a,b)\}$ is then a measurable neighborhood of (a_o,b_o), by (5), hence has positive λ-measure. Thus (3.14) ~~(14) of section 3~~ is false. This proves that (3.14) ~~(14) of section 3~~ implies (1).

In variants I and III, (2) is vacuously true. In variants II and IV let (2) be false, so that $(\mu' - \lambda')$ has a point of support a_o for which $p(a_o) > 0$. By (6), $\{a\,|\,p(a) > 0\}$ is a measurable neighborhood of a_o, hence has positive $\mu' - \lambda'$ measure. Thus (3.15) ~~(15) of section 3~~ is false; ~~and~~ It follows that (3.15) ~~(15) of section 3~~ implies (2) in all cases.

Finally, (3) is vacuously true in variants I and II. In variants III and IV let (3) be false, and conclude by an argument similar to that just given that (3.16) ~~(16) of section 3~~ is false. Thus (3.16) ~~(16) of section 3~~ implies (3). ~~This concludes the proof.~~ |||

Note that in variant I (5) alone suffices to insure that a measure potential is a topological potential, and in variants II and III only one extra condition is needed.

Potentials have been defined in terms of a pair of functions, (p,q). It often happens, however — as in ~~the next~~ chapter 9 — that one of these functions arises naturally from the problem situation and has a natural interpretation, while the other does not. For this reason it is useful to have a concept involving just one function. Suppose, then, one is given the ingredients of a transportation problem: measure spaces (A,Σ',μ'), (B,Σ'',μ''), measurable cost function $f:A \times B \rightarrow$ reals, with topologies T', T'' on A, B, respectively. Let λ be a feasible flow. Measurable function $p:A \rightarrow$ reals is a <u>left half-potential</u> for λ iff

$$p(a_o) + f(a_o,b_o) \leq p(a) + f(a,b_o) \qquad (7.5.8)$$

for all $a,a_o \in A$, $b_o \in B$ such that (a_o,b_o) is a point of support for λ (relative to the topology $T' \times T''$). That is, for fixed

$\underline{b}_o$, the function $\underline{p}(\cdot) + \underline{f}(\cdot,\underline{b}_o)$ attains its infimum at any point $\underline{a}_o \in \underline{A}$ ~~which~~ that, paired with $\underline{b}_o$, supports λ. Similarly, measurable function $\underline{q}:\underline{B} \to$ reals is a right half-potential for λ iff

$$\underline{q}(\underline{b}_o) - \underline{f}(\underline{a}_o,\underline{b}_o) \geq \underline{q}(\underline{b}) - \underline{f}(\underline{a}_o,\underline{b}) \qquad (7.5.9)\ (9)$$

for all $\underline{a}_o \in \underline{A}$, $\underline{b},\underline{b}_o \in \underline{B}$ such that $(\underline{a}_o,\underline{b}_o)$ supports λ.

One easily verifies that, if $(\underline{p},\underline{q})$ is a topological potential for λ, then $\underline{p}$ and $\underline{q}$ separately are left and right half-potentials for λ, respectively. Indeed, from (3.3) ~~(3) of section 3~~ and (1) we obtain

$$\underline{p}(\underline{a}_o) + \underline{f}(\underline{a}_o,\underline{b}_o) = \underline{q}(\underline{b}_o) \leq \underline{p}(\underline{a}) + \underline{f}(\underline{a},\underline{b}_o)$$

whenever $(\underline{a}_o,\underline{b}_o)$ supports λ. This yields (8), and a similar argument yields (9). Conversely, given a half-potential, ~~one~~ we can lay down certain conditions under which the opposite half-potential exists, the two together being a potential $(\underline{p},\underline{q})$. (Thus, if $\underline{p}$ is given, $\underline{q}$ might be defined by: $\underline{q}(\underline{b}) = \inf\{\underline{p}(\underline{a}) + \underline{f}(\underline{a},\underline{b})\}$, the infimum taken over all $\underline{a} \in \underline{A}$). The following theorem, ~~together~~ with the rest of this section, accomplishes this task indirectly, and at the same time relates these to another concept of interest.

Theorem: Let flow λ have a half-potential, and let $(\underline{a}_i,\underline{b}_i)$, $(\underline{i} = 1,\ldots,\underline{n})$ be points of support for λ $(\underline{n} \geq 2)$. Then

$$\underline{f}(\underline{a}_1,\underline{b}_1) + \ldots + \underline{f}(\underline{a}_n,\underline{b}_n) \leq \underline{f}(\underline{a}_1,\underline{b}_2) + \ldots + \underline{f}(\underline{a}_{n-1},\underline{b}_n) + \underline{f}(\underline{a}_n,\underline{b}_1). \qquad (7.5.10)\ (10)$$

Proof: Suppose λ has a left half-potential p. Then

$$p(a_i) + f(a_i,b_i) \leq p(a_{i-1}) + f(a_{i-1},b_i)$$

~~for~~ $i = 2,\ldots,n$, and also

$$p(a_1) + f(a_1,b_1) \leq p(a_n) + f(a_n,b_1),$$

all from (8). Adding these n inequalities, the p's drop out, and we are left with (10). The proof for a right half-potential is similar. |||

Inequality (10) is sufficiently interesting to merit a name. ~~Let us~~ call it the circulation condition. Intuitively, it says the following. Suppose we cyclically reassign sources to sinks, shifting some outflow from a_1 away from b_1 and to b_2, etc., and completing the circle by shifting a_n-outflow from b_n to b_1. This reassignment — which leaves all total source outflows and sink inflows unaltered — does not reduce total cost, according to (10).

For $n = 2$ in particular, the circulation condition bears a striking resemblance to the concept of comparative advantage. (Think of a_1 and a_2 as two countries or workers, and of b_1 and b_2 as alternative activities in which they can engage). Comparative advantage is usually expressed as an inequality among products or ratios, however, while the circulation condition is an inequality among sums or differences.

We now come to the demonstration of the existence of topological potentials. This goes through two main stages, each rather long. We start with λ^{o}, an optimal solution to the transportation problem; more precisely, λ^{o} is unsurpassed under (reverse) standard ordering of pseudomeasures. From this we deduce five inequalities involving the cost function f. The crucial one is exactly the circulation condition, (11) below. This is used to establish the existence and major properties of the topological potential. The other inequalities are needed for (2), (3), and non-negativity conditions on p and q.

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From now on we use the abbreviation "ab" for $f(a,b)$.

Lemma: Let (A,Σ',μ') and (B,Σ'',μ'') be sigma-finite measure spaces, and $f: A \times B \to$ reals measurable; let T', T'' be topologies on A, B, respectively, such that $T' \subseteq \Sigma'$, $T'' \subseteq \Sigma''$[13], and f is continuous with respect to $T' \times T''$. Let λ^{o} be unsurpassed (reverse standard order) for the transportation problem formed from the above, and let (a_i, b_i) $(i = 1,\ldots,n)$ be points of support for λ^{o}. Then

(i) $$a_1b_1 + \ldots + a_nb_n \leq a_1b_2 + \ldots + a_{n-1}b_n + a_nb_1 \qquad \text{(7.5.11)} \quad \cancel{(11)}$$

holds for all problem variants, I, II, III, IV $(n = 2,3,\ldots)$.

(ii) If $a_\theta \in A$ is a point of support for $(\mu' - \lambda^{o\prime})$, then, in variants II and IV we have

$$a_1b_1 + \ldots + a_nb_n \leq a_\theta b_1 + \ldots + a_{n-1}b_n \qquad \text{(7.5.12)} \quad (12)$$

$(n = 1,2,\ldots)$, and in variant IV we also have

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$$a_1b_1 + \ldots + a_nb_n \le a_0b_1 + \ldots + a_nb_{n+1} \qquad (7.5.13)\ \ (13)$$

$(n = 0,1,\ldots)$. (For $n=0$ the left side of (13) is zero.)

(iii) If $b_1 \in B$ is a point of support for $\lambda^{o''} - \mu''$, and all subsets of B belong to $\mathcal{T}''$ (hence to Σ''), then

$$a_1b_1 + \ldots + a_nb_n \le a_1b_2 + \ldots + a_nb_{n+1} \qquad (7.5.14)\ \ (14)$$

$(n = 1,2,\ldots)$, in variants III and IV, and

$$a_1b_1 + \ldots + a_nb_n \le a_1b_2 + \ldots + a_{n-1}b_n \qquad (7.5.15)\ \ (15)$$

$(n = 1,2,\ldots)$ in variant IV. (For $n=1$ the right side of (15) is zero.) (Neither the a_i's nor the b_i's need to be ~~all~~ distinct in any of these cases).

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Proof: For each case, (11)–(15), we construct a new feasible flow; the corresponding inequality is then deduced from the fact that this new flow cannot surpass λ^o. Only (11) will be proved in detail.

Given $\varepsilon > 0$, there are open sets $E_1,\ldots,E_n \subseteq A$ and $F_1,\ldots,F_n \subseteq B$, satisfying $a_i \in E_i$, $b_i \in F_i$ $(i = 1,\ldots,n)$; and, for all $a \in E_i$, $b \in F_i$,

$$|ab - a_ib_i| \le \varepsilon \qquad (7.5.16)\ \ (16)$$

$(i = 1,\ldots,n)$; and, for all $a \in E_i$, $b \in F_{i+1}$,

$$|ab - a_ib_{i+1}| \le \varepsilon \qquad (7.5.17)\ \ (17)$$

$(i = 1,\ldots,n)$. (In (17) for $i = n$, F_{n+1} and b_{n+1} are to be understood as F_1, b_1, respectively). For, the continuity of f

implies ~~that~~ there are open sets owning a_i and b_i, respectively, such that (16) is satisfied for all (a,b) in their cartesian product, and there are open sets owning a_i and b_{i+1} such that (17) is similarly satisfied. This gives two open sets for each of the points $(a_1, \ldots, a_n, b_1, \ldots, b_n)$. Let E_i be the intersection of the two open sets for a_i, and construct F_i in the same way for b_i $(i = 1, \ldots, n)$. With these, all the relations above are satisfied simultaneously.

then $\lambda^\circ(E_i \times F_i) > 0$, since $E_i \times F_i$ is a measurable neighborhood of (a_i, b_i), a point of support for λ°. $\lambda^\circ(E_i \times F_i)$ may in fact be infinite; but if so there is a measurable set $G_i \subseteq E_i \times F_i$ such that $\infty > \lambda^\circ(G_i) > 0$, since λ° is ~~sigma~~ σ-finite. Choose a set G_i satisfying these conditions for each $i = 1, \ldots, n$.

Now define measures ν_i on $(A \times B, \Sigma' \times \Sigma'')$ as follows: set

$$\nu_i(H) = \lambda^\circ(H \cap G_i)/\lambda^\circ(G_i), \qquad (7.5.18)\ \text{(18)}$$

$H \in \Sigma' \times \Sigma''$, $i = 1, \ldots, n$. Consider the signed measure ν given by:

$$(\nu_1' \times \nu_2'') + \ldots + (\nu_{n-1}' \times \nu_n'') + (\nu_n' \times \nu_1'') - \nu_1 - \ldots - \nu_n. \qquad (7.5.19)\ \text{(19)}$$

(Here ν_1' is the left marginal of ν_1, and ν_2'' is the right marginal of ν_2; we are to form the product measure of these, add up n similar product measures, and then subtract the ν_i's of (18). All the summands in (19) have universe set $A \times B$, of course).

For each $\underline{i}$, $\nu_{\underline{i}}(\underline{A} \times \underline{B}) = 1$, so that (19) is well-defined, and in fact bounded. Finally, consider the (signed) measure λ given by

$$\lambda = \lambda^{o} + \underline{y}\nu, \tag{20}$$

(7.5.20)

where $\underline{y}$ is the positive real number $\frac{1}{n}\min\{\lambda^{o}(G_{\underline{i}}) | \underline{i} = 1,\ldots,\underline{n}\}$. no ¶

22-11 ¶ We ~~shall~~ prove that this λ is a feasible flow. First, λ is non-negative. For, $\underline{y}\nu_{\underline{i}}(\underline{H}) \leq \lambda^{o}(\underline{H})/\underline{n}$, all $\underline{i}$, from the definition of $\underline{y}$, so that

$$\lambda(\underline{H}) \geq \lambda^{o}(\underline{H}) - \underline{y}\nu_1(\underline{H}) - \ldots - \underline{y}\nu_{\underline{n}}(\underline{H}) \geq 0.$$

Next, for any $\underline{E} \in \Sigma'$ we obtain $\nu(\underline{E} \times \underline{B}) = 0$ by direct substitution in (19). Similarly, $\nu(\underline{A} \times \underline{F}) = 0$ for any $\underline{F} \in \Sigma''$. This means that λ^{o} and λ have the same marginals. Hence, in any variant of the transportation problem, the feasibility of λ^{o} implies the feasibility of λ.

It follows that λ cannot (downwardly) surpass λ^{o}. Now

$$\int \hat{f}\, d\lambda - \int \hat{f}\, d\lambda^{o} = \hat{y} \int \hat{f}\, d\nu.$$

But the indefinite integral $\int \hat{f}\, d\nu$ is actually well-defined and finite as a definite integral over $\underline{A} \times \underline{B}$. For, $\underline{f}$ is bounded on the set

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$$(E_1 \times F_1) \cup \ldots \cup (E_{\underline{n}} \times F_{\underline{n}}) \cup (E_1 \times F_2) \cup \ldots \cup (E_{\underline{n}} \times F_1)$$

by (16) and (17), while ν is zero off this set. Hence unsurpassedness under (reverse) standard ordering reduces to the set

condition

$$\int_{A\times B} f\, d\nu \geq 0. \qquad \text{(7.5.21)}\ \cancel{(21)}$$

Expanding this by (19), we find that

$$\int_{A\times B} f\, d\nu_i \geq (a_ib_i - \varepsilon)\nu_i(A \times B) = (a_ib_i - \varepsilon) \qquad \text{(7.5.22)}\ \cancel{(22)}$$

($i = 1,\ldots,n$), since ν_i is zero off $E_i \times F_i$, and f is bounded below by $a_ib_i - \varepsilon$ on $E_i \times F_i$, according to (16). Similarly,

$$\int_{A\times B} f\, d(\nu_i' \times \nu_{i+1}'') \leq (a_ib_{i+1} + \varepsilon)\nu_i'(A)\nu_{i+1}''(B) = (a_ib_{i+1} + \varepsilon), \qquad \text{(7.5.23)}\ \cancel{(23)}$$

($i = 1,\ldots,n$), since $f \leq a_ib_{i+1} + \varepsilon$ on $E_i \times F_{i+1}$, by (17), while $\nu_i' \times \nu_{i+1}''$ is zero off this set. (For $i = n$, $i + 1$ should be read as "1").

From ~~(21), (22), and (23)~~ (21)–(23) we obtain

$$a_1b_2 + \ldots + a_nb_1 - a_1b_1 - \ldots - a_nb_n \geq -2n\varepsilon. \qquad \text{(7.5.24)}\ \cancel{(24)}$$

But ε is an arbitrary positive number. Hence (24) implies the circulation condition (11). The first inequality has been obtained.

We now sketch the proof of (12) and (13). It proceeds as above with the following differences. In addition to the open sets E_i, F_i ($i = 1,\ldots,n$), we find open sets $E_0 \subseteq A$ and — for (13) — $F_{n+1} \subseteq B$, such that $a_0 \in E_0$, $b_{n+1} \in F_{n+1}$, and (17) holds for $i = 0,1,\ldots,n$.

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The sets $G_i \subseteq (E_i \times F_i)$, $(i = 1,\ldots,n)$ are constructed as above. In addition we find that $(\mu' - \lambda^{o\prime})(E_\theta) > 0$, since E_θ is a measurable neighborhood of a_θ, a point of support for $\mu' - \lambda^{o\prime}$. $(\mu' - \lambda^{o\prime})(E_\theta)$ may in fact be infinite, but there is always a measurable set $A' \subseteq E_\theta$ for which $\infty > (\mu' - \lambda^{o\prime})(A') > 0$, since these measures are ~~sigma~~ σ-finite.

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The measures ν_i, $(i = 1,\ldots,n)$, are defined exactly as in (18). In addition, define the measure μ'_θ on (A,Σ') by

$$\mu'_\theta(E) = [(\mu' - \lambda^{o\prime})(E \cap A')]/(\mu' - \lambda^{o\prime})(A'),$$

all $E \in \Sigma'$. Also — for (13) — define measure μ''_{n+1} on (B,Σ'') by

$$\mu''_{n+1}(F) = 1 \quad \text{if } b_{n+1} \in F; = 0 \text{ if } b_{n+1} \notin F,$$

all $F \in \Sigma''$. Consider the signed measure ν on $(A \times B,\ \Sigma' \times \Sigma'')$ given by:

$$(\mu'_\theta \times \nu''_1) + (\nu'_1 \times \nu''_2) + \ldots$$

$$+ (\nu'_{n-1} \times \nu''_n) - \nu_1 - \ldots - \nu_n[+(\nu'_n \times \mu''_{n+1})], \qquad \text{(7.5.25)} \ ~~(25)~~$$

(This formula applies for $n \geq 1$. For $n = 0$, which arises only for (13), define ν as $\mu'_\theta \times \mu''_1$).

where the bracketed measure is to be included when considering (13), and omitted when considering (12). Each of the measures in (25) has the value one at $A \times B$, so (25) is well-defined and bounded. Finally, consider the (signed) measure λ given by

$$\lambda^o + z\nu \qquad \text{(7.5.26)} \ ~~(26)~~$$

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where $\underline{z}$ is ~~now~~ the positive real number $\underline{\min}\left(\underline{y},\ (\mu'-\lambda^{\circ\prime})(\underline{A}')\right)$, $\underline{y}$ being defined as in (20) above.

Non-negativity of λ is proved as above. Next, substitute $\underline{E} \times \underline{B}$ into (25) (<u>inclusive</u> of the bracketed term), where $\underline{E} \in \Sigma'$. The result is $\mu'_\theta(\underline{E})$. Hence,

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$$\lambda'(\underline{E}) = \lambda^{\circ\prime}(\underline{E}) + z\mu'_\theta(\underline{E}) \leq \lambda^{\circ\prime}(\underline{E}) + (\mu' - \lambda^{\circ\prime})(\underline{E}),$$

the inequality arising from the definition of $\underline{z}$. Hence $\lambda' \leq \mu'$: λ satisfies the capacity constraints in variants <u>II</u> and <u>IV</u>. The same result holds <u>a fortiori</u> if the bracketed term in (25) is omitted, since this leads to a smaller λ.

Next, substitute $\underline{A} \times \underline{F}$ into (25) (<u>omitting</u> the bracketed term), where $\underline{F} \in \Sigma''$. Everything cancels, hence $\lambda'' = \lambda^{\circ\prime\prime}$ and feasibility is preserved for the requirement constraints in any variant. Adding in the bracketed term, however, yields merely $\lambda'' \geq \lambda^{\circ\prime\prime}$, so that feasibility is preserved in variant <u>IV</u>. To summarize: λ given by (26) is feasible for variants <u>II</u> and <u>IV</u> if ν is defined by (25) <u>omitting</u> the bracketed term, and is feasible for <u>IV</u> if ν <u>includes</u> the bracketed term.

Just as above, the feasibility of λ leads to $\int_{\underline{A}\times\underline{B}} f\, d\nu \geq 0$, and an argument similar to the one above gives the inequalities (12) or (13), depending on whether (25) omits or includes the bracketed term, respectively. These inequalities are then valid for these variants in which λ is feasible.

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18 Finally, the proof for (14)–(15) is very similar to that for (12)–(13). There is no point a_θ, and consequently no E_θ, A', or μ'_θ. We choose F_1 to be the singleton set $\{b_1\}$, which can be done because every subset of B is open by assumption. Otherwise proceeding exactly as above, we define ν by (25), modified by the omission of the measure $\mu'_\theta \times \nu''_1$. The bracketed measure $\nu'_n \times \mu''_{n+1}$ is to be included when considering (14), and omitted when considering (15). λ is again defined by (26), where z, however, is now the positive real number

$$\min\left(y,\ \lambda^{\circ\prime\prime}\{b_1\} - \mu''\{b_1\}\right), \qquad (7.5.27)\ \cancel{(27)}$$

y being defined as in (20). To show that z is indeed positive, note that $\{b_1\}$ is itself a measurable neighborhood of b_1, hence has positive $(\lambda^{\circ\prime\prime} - \mu'')$-measure since b_1 is a point of support for that measure.

Non-negativity of λ is proved as above. Next, substitute $E \times B$ into the modified (25) (<u>inclusive</u> of the bracketed term), where $E \in \Sigma'$. Everything cancels, hence $\lambda' = \lambda^{\circ\prime}$ and feasibility is preserved for the ~~requirement~~ capacity constraints in any variant. Omitting the bracketed term yields $\lambda' \leq \lambda^{\circ\prime}$, which preserves feasibility in variant IV.

Next, substitute $A \times F$ into the modified (25) (<u>omitting</u> the bracketed term), where $F \in \Sigma''$. The result is $-\nu_1(A \times F)$. Now, since $F_1 = \{b_1\}$, we have $G_1 \subseteq E_1 \times \{b_1\}$. Hence, from (18), with $i = 1$, we find that $\nu_1(A \times F) = 1$ if $b_1 \in F$, and $\nu_1(A \times F) = 0$

if $\underline{b}_1 \notin \underline{F}$. In this second case we have $\lambda''(\underline{F}) = \lambda^{\circ\prime\prime}(\underline{F})$, while on $\{\underline{b}_1\}$ we have

$$\lambda''\{\underline{b}_1\} = \lambda^{\circ\prime\prime}\{\underline{b}_1\} - \underline{z} \geq \mu''\{\underline{b}_1\},$$

from (27). Hence $\lambda'' \geq \mu''$, and λ is feasible for the requirement constraints in variants III and IV. Adding in the bracketed term of (25) can only increase λ, and hence preserves this feasibility. ~~To~~ We summarize: if the bracketed measure in modified (25) is included, the λ thus defined is feasible for variants III and IV; if it is omitted, λ is still feasible for IV.

The argument from unsurpassedness to (14) or (15) then follows the pattern laid down above. ||| □

We now come to the main result. ~~Note that~~ we prove the existence of a topological potential in the wide sense, ~~— that is,~~ i.e., the integrals $\int_{\underline{A}} \underline{p}\, d\mu'$, $\int_{\underline{B}} \underline{q}\, d\mu''$ need not be finite, or even well-defined. As mentioned below, a simple extra premise removes this qualification. Note ~~also~~ that the premises for variants III and IV are somewhat stronger than for variants I and II.

Theorem: Let $(\underline{A},\Sigma',\mu')$ and $(\underline{B},\Sigma'',\mu'')$ be non-empty sigma-finite measure spaces, and let $\underline{f}:\underline{A} \times \underline{B} \to$ reals be measurable and bounded. Let λ° be a measure on $(\underline{A} \times \underline{B},\ \Sigma' \times \Sigma'')$ ~~which~~ that is unsurpassed (reverse standard order) for the transportation problem determined by these.

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Also let $\mathcal{T}'$, $\mathcal{T}''$ be topologies on A, B, respectively, such that $\mathcal{T}' \subseteq \Sigma'$, $\mathcal{T}'' \subseteq \Sigma''$, and f is continuous with respect to $\mathcal{T}' \times \mathcal{T}''$. (In variants III and IV make the additional assumption that all subsets of B belong to $\mathcal{T}''$).

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Then there exist bounded functions, p:A → reals, q:B → reals, p lower and q upper semi-continuous, such that (p,q) is a topological potential for λ° in the wide sense.

Proof: We ~~shall~~ use different definitions of p for the different variants I, II, III, IV. An α-sequence is a sequence of the form $a_0, b_1, a_1, \ldots, b_n, a_n$ (n = 0 or 1 or ...), where (a_i, b_i) is a point of support of λ° for i = 1,...,n. That is, it consists of 2n+1 points, alternating from A and B, beginning and ending with an A-point. (For n = 0 the sequence consists of a single A-point, a_0). A β-sequence is an α-sequence with an extra B-point b_{n+1} tacked on at the end. (Thus the shortest β-sequence is of the form a_0, b_1).

The value of an α- (β-) sequence $a_0, b_1, \ldots, a_n, (b_{n+1})$ is defined as

$$-a_0b_1 + a_1b_1 - a_1b_2 + \ldots + a_nb_n \; (-a_nb_{n+1}), \qquad (28)$$

(7.5.28)

where the parenthetical term is included for β-sequences only. (Here "ab" abbreviates f(a,b) as usual. The singleton sequence a_0 is taken to have the value zero).

Now define the function p, with domain A, in three ways. The α-definition sets p(a) equal to the supremum of the values of all α-sequences beginning with a, all $a \in A$. The β-definition is the same with all β-sequences beginning with a. The γ-definition is the maximum of these two, i.e., ~~in other words,~~ the supremum of the values of all α- and β-sequences beginning with a.

We now prove that, under any of these definitions, ~~the function~~ p is bounded, lower semi-continuous, and measurable. Function f is bounded, so $|f| \leq N$ for some real number N. Hence $p(a) \geq 0$ or $-ab$ on the α- or β-definition, respectively, for any $a \in A$, $b \in B$, so that $p \geq -N$: p is bounded below.

Next, we prove boundedness above. Sequences of length a~~t~~ most four have values ~~which are~~ not greater than 3N. For sequences of length five or more we make use of the circulation condition (11), which is valid in all variants under our premises. Formula (28) may be rewritten as

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$$[a_1b_1 - a_1b_2 + \ldots - a_{n-1}b_n + a_nb_n - a_nb_1] + a_nb_1 - a_0b_1 \; (-a_nb_{n+1}) \qquad (29)$$

(7.5.29)

($n = 2,3,\ldots$). But (11) states that the bracketed expression is non-positive. Hence (29) never exceeds 3N. Thus p is bounded.

Lower semi-continuity is proved as follows. Think of all points in (28) except a_0 as being fixed. (28) then defines a family of real-valued functions with common domain A, each sequence $b_1, a_1, \ldots, b_n, a_n, (b_{n+1})$ indexing one such function.

Each of these functions is continuous, since $\underline{f}$ is continuous. One easily shows that the supremum of a set of continuous functions is lower semi-continuous. But $\underline{p}$ on any definition is such a supremum.

The sets $\{a|p(a) > x\}$, x real, are open, by lower semi-continuity. But $T' \subseteq \Sigma'$, hence these sets are also measurable. Thus p is measurable. (Similarly, an upper semi-continuous function on B is measurable. This fact is needed later).

With these general results in hand, we proceed to each variant in turn.

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Variant I. Use any of the three definitions of p, ~~and~~ then define $q:B \to$ reals by

$$q(b) = \inf[p(a) + ab], \qquad (7.5.30)\ \cancel{(30)}$$

the infimum taken over all points $a \in A$. Since p and f are bounded, q is bounded. Think of $p(a) + ab$ as a family of functions of $\underline{b}$, indexed by $\underline{a}$. Each of these functions is continuous since f is continuous. Then q, as the infimum of a set of continuous functions, is upper semi-continuous, hence also measurable.

It remains to show that (p,q) is a topological potential for λ^o in the wide sense. For variant I this reduces to ~~the following:~~

$$q(b) - p(a) \leq ab \qquad (7.5.31)\ \cancel{(31)}$$

for all $a \in A$, $b \in B$, with equality if (a,b) supports λ^o. (However, (31) ~~however,~~ is an immediate consequence of ~~definition~~ (30). Let (a_0,b_0) support λ^o. Then

$a_0 b_0$ term

$$\underline{p}(\underline{a}) \geq -\underline{a}\underline{b}_0 + \underline{a}_0\underline{b}_0 + \underline{p}(\underline{a}_0) \qquad (7.5.32) \quad \cancel{(32)}$$

for any $\underline{a} \in \underline{A}$; For, the right side of (32) is the supremum of the values of sequences (α-, β-, or both) beginning $(\underline{a}, \underline{b}_0, \underline{a}_0, \ldots)$, while the left is the supremum over a wider class of such sequences. It follows that the infimum of $\underline{p}(\underline{a}) + \underline{a}\underline{b}_0$ is attained at $\underline{a} = \underline{a}_0$. Hence

hyphens

$$\underline{q}(\underline{b}_0) = \underline{p}(\underline{a}_0) + \underline{a}_0\underline{b}_0, \qquad (7.5.33) \quad \cancel{(33)}$$

so that (31) is satisfied with equality. ~~This completes the proof for variant I.~~

#

Variant II. Use the α-definition of $\underline{p}$, and define $\underline{q}$ again by (30). All of the argument for variant $\underline{I}$ applies here, also ~~too~~. To complete the proof for variant $\underline{II}$, two additional facts must be established: $\underline{p} \geq 0$, and $\underline{p}(\underline{a}_0) = 0$ for any point $\underline{a}_0$ supporting $\mu' - \lambda^{\circ\prime}$.

For any $\underline{a} \in \underline{A}$, the singleton consisting of $\underline{a}$ alone is an α-sequence of value zero; hence $\underline{p} \geq 0$. Let $\underline{a}_0$ support $\mu' - \lambda^{\circ\prime}$. Then inequality (12) is valid, and states that (28) — omitting the parenthetical term — is never positive; hence $\underline{p}(\underline{a}_0) = 0$. ~~This finishes variant II.~~

#

Variant III. Use the β-definition of $\underline{p}$, and define $\underline{q}(\underline{b})$ by (30) if $\underline{b}$ is $\underline{\text{not}}$ a point of support for $\lambda^{\circ\prime\prime} - \mu''$, while $\underline{q}(\underline{b}) = 0$ if $\underline{b}$ $\underline{\text{is}}$ a point of support for $\lambda^{\circ\prime\prime} - \mu''$.

Function
q is clearly bounded. Since T'', hence Σ'', own all subsets of B, q is automatically continuous and measurable. To show that (p,q) is a topological potential for λ^o in the wide sense we must demonstrate (31) (with equality if (a,b) supports λ^o), and also demonstrate that $q \geq 0$, with $q(b_0) = 0$ for all points of support of $\lambda^{o''} - \mu''$. This last fact is true by the definition of q.

For any $a \in A$, $b \in B$ we have $p(a) \geq -ab$, since the latter is the value of the β-sequence a,b; hence $q \geq 0$. The same inequality yields (31) on the support of $\lambda^{o''} - \mu''$, since $q = 0$ there. (31) follows from (30) off the support.

Finally, let (a_0, b_0) support λ^o. If $q(b_0)$ satisfies (30) the argument of (32) leads to the equality (33). This leaves the case where b_0 supports $\lambda^{o''} - \mu''$. Then inequality (14) is valid, in the form:

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$$-a_0b_0 \geq -a_0b_1 + a_1b_1 - a_1b_2 + \ldots - a_nb_{n+1} \tag{34}$$

(7.5.34)

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$(n = 0,1,\ldots)$. Here $b_1, a_1, \ldots, a_n, b_{n+1}$ are any points such that (a_i, b_i) supports λ^o $(i = 1,\ldots,n)$. But, by (28), $p(a_0)$ is the supremum of the right-hand expression in (34) over all such sequences. Hence $p(a_0) \leq -a_0b_0$. Since the opposite inequality always holds, $p(a_0) = -a_0b_0$. But this is (33), since $q(b_0) = 0$. Hence (31) holds with equality if (a,b) supports λ^o. ~~This finishes variant III.~~

II
4 1/8

Variant IV: Use the γ-definition of p, and define q as in variant III. The same arguments as in variant III then show that q is bounded, continuous, measurable, non-negative, and zero on the support of $\lambda^{o}{}'' - \mu''$, and that (31) holds for all $a \in A$, $b \in B$. Three more facts need to be demonstrated: $p \geq 0$, $p = 0$ on the support of $\mu' - \lambda^{o}{}'$, and (33) holds for (a_0, b_0) supporting λ^{o}.

For any $a \in A$, $p(a)$ is not less than the value of the singleton sequence a; hence $p \geq 0$. Next, let a_0 support $\mu' - \lambda^{o}{}'$. Then inequalities (12) and (13) are valid, and imply that (28) never exceeds zero, with or without the parenthetical term; hence $p(a_0) = 0$.

Finally, let (a_0, b_0) support λ^{o}. As above, if b_0 does not support $\lambda^{o}{}'' - \mu''$, then $q(b_0)$ satisfies (30) and the argument of (32) yields (33). Suppose b_0 does support $\lambda^{o}{}'' - \mu''$. Then inequalities (14) and (15) are valid, in the form:

$$-a_0b_0 \geq -a_0b_1 + a_1b_1 - \ldots + a_nb_n(-a_nb_{n+1}) \qquad (35)$$

(7.5.35)

($n = 0,1,\ldots$), with or without the parenthetical term. (Note that for $n = 0$, (15) in the form (35) is simply: $-a_0b_0 \geq 0$). By (28), $p(a_0)$ on definition γ is the supremum of all the right-hand expressions in (35), for $b_1, a_1, \ldots, b_n, a_n, (b_{n+1})$, where (a_i, b_i) supports λ^{o} $(i = 1,\ldots,n)$. Hence again $p(a_0) \leq -a_0b_0$, and the same argument as in variant III shows that (33) is satisfied. This finishes variant IV and the proof. ■

As noted above, (p,q) is a topological potential in the wide sense. But if we impose the additional premise that μ'

and μ'' are bounded, ~~then~~ (p,q) is ~~in fact~~ a potential in the strict sense. For, the functions p and q as constructed above are also bounded, hence the definite integrals, $\int_A p\, d\mu'$ and $\int_B q\, d\mu''$, are both well-defined and finite.

The extra premise that all subsets of B belong to T'' — imposed for variants III and IV — is somewhat restrictive (although natural in some cases, e.g., when B is a finite set). In variant IV it may be replaced by the extra premises that f is positive and $\lambda^{\circ\prime\prime}$ ~~sigma~~-finite.[14] We show this as follows. Using the Radon-Nikodym theorem as in case (IVb) of the optimality existence theorem, we show the existence of a measure λ_1 on $\Sigma' \times \Sigma''$ satisfying $\lambda_1 \leq \lambda^{\circ}$ and being feasible for variant II, hence for the original variant IV. Since λ_1 does not surpass λ°, and $f > 0$, we must in fact have $\lambda_1 = \lambda^{\circ}$. Thus λ° is feasible for variant II, and in fact unsurpassed for it. Let (p,q) be a wide-sense topological potential for λ° constructed as for variant II. To verify that this (p,q) is also a variant IV wide-sense potential, we must demonstrate two additional facts: $q \geq 0$, and $q = 0$ on the support of $\lambda^{\circ\prime\prime} - \mu''$. This last property is trivial, because the support is empty, since $\lambda^{\circ\prime\prime} = \mu''$. Also q is defined by (30), so the non-negativity of p and f imply the non-negativity of q. This concludes the proof.

The functions p and q constructed in our main theorem are semi-continuous. We can strengthen this result by adding some extra premises concerning f. First, we need a few continuity concepts.

Let $(C,\mathcal{T})$ be a topological space, and $\mathcal{G}$ a set of real-valued functions with common domain C: set $\mathcal{G}$ is equicontinuous at the point $c_0 \in C$ iff, for every positive number ε there is a neighborhood N of c_0 such that

$$|g(c) - g(c_0)| < \varepsilon, \qquad \text{(7.5.36)} \ ~~(36)~~$$

for all $c \in N$ and all $g \in \mathcal{G}$. (If $\mathcal{G}$ consists of just one function g, this is simply the definition of continuity of g at the point c_0. One may then show that g is continuous as defined previously iff it is continuous at every point of its domain as defined by (36).) Next, suppose that $\mathcal{T}$ is generated by a metric d. Set $\mathcal{G}$ is uniformly equicontinuous iff, for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\text{if } d(c_1, c_2) < \delta, \text{ then } |g(c_1) - g(c_2)| < \varepsilon, \qquad \text{(7.5.37)} \ ~~(37)~~$$

for all $c_1, c_2 \in C$ and all $g \in \mathcal{G}$. (If $\mathcal{G}$ consists of just one function g, this is simply the definition of uniform continuity of g).

Now consider the transport cost function $f: A \times B \to$ reals. f may be thought of as a family of functions $f(\cdot, b): A \to$ reals, indexed by $b \in B$. Suppose this family ~~were~~ is equicontinuous at the point $a' \in A$, ~~That is~~ i.e., for all $\varepsilon > 0$ there is a neighborhood N of a' such that

$$|f(a,b) - f(a',b)| < \varepsilon$$

Start

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for all $a \in N$, $b \in B$. Then we claim that p, constructed under any of the definitions α, β, γ, is continuous at a'.

The proof of this statement rests on the easily-verified fact that, if the collection of functions $\mathcal{G}$ satisfies (36), and the supremum of $\mathcal{G}$ is ~~real-valued~~ finite, then sup $\mathcal{G}$ is continuous at c_0. Now consider (28) as a function of a_0, the other points $b_1, a_1, \ldots, b_n, a_n, (b_{n+1})$ being parameters. This function differs only by a constant from the function $-f(\cdot, b_1)$. Hence the functions (28) are equicontinuous at a', so that p, which is the supremum of an appropriate subset of them, is continuous at a'. This concludes the proof.

Thus, if the family $f(\cdot, b)$, $b \in B$, is equicontinuous at each point $a \in A$, it follows that p is continuous, not merely lower semi-continuous.

What about q? Function f may also be thought of as a family of functions $f(a, \cdot)$ with common domain B, indexed by $a \in A$. Suppose this family ~~were~~ is equicontinuous at $b'' \in B$. Then we claim that q, given by (30), is continuous at b''.

The proof is virtually the same as that for p, on noting that, if $\mathcal{G}$ satisfies (36), then the infimum of $\mathcal{G}$ is also continuous at c_0, provided inf $\mathcal{G}$ is ~~real-valued~~ finite. (This takes care of variants I and II; in variants III and IV all subsets of B belong to $\mathcal{T}''$, which makes any function $q: B \to$ reals continuous).

Next, suppose $\mathcal{T}'$, the topology of A, is generated by a metric d', under which the family $f(\cdot, b)$, $b \in B$, is uniformly

<u>equicontinuous</u>. Then p is <u>uniformly continuous</u>, under any of the definitions α, β, γ.

The proof is similar to those above, and is based on the fact — again easily verified — that if the family $\mathcal{G}$ satisfies (37), and sup $\mathcal{G}$ is ~~real-valued~~ finite, then sup $\mathcal{G}$ is uniformly continuous. Reversing the rôles of A and B, we get a similar condition implying that q (given by (30)) is uniformly continuous.

Our final theorem summarizes many of the preceding results. ~~Note that~~ the boundedness assumptions guarantee that the objective function for the primal is always a finite definite integral. Hence we speak merely of an optimal solution, the distinction between "best" and "unsurpassed" disappearing in the present instance.

D

<u>Theorem</u>: Let (A, Σ', μ') and (B, Σ'', μ'') be bounded measure spaces, and let $f: A \times B \to$ reals be bounded measurable. Let λ° be an optimal flow for the transportation problem determined by these.

Also let T', T'' be topologies on A, B, respectively, such that $T' \subseteq \Sigma'$, $T'' \subseteq \Sigma''$, f is continuous with respect to the product topology $T' \times T''$, and the latter has the strong Lindelöf property. (In variants III and IV, add the condition that all subsets of B belong to T'').

Then ~~there exist~~ functions $p^\circ: A \to$ reals, $q^\circ: B \to$ reals exist such that the pair (p°, q°) is best for the dual of the transportation problem, and the primal and dual values are equal:

1.5 type

Stat 23-5 100]

$$\int_{A\times B} f \, d\lambda^{o} = \int_{B} q^{o} \, d\mu'' - \int_{A} p^{o} \, d\mu' .$$

Proof: By the main result of this section ~~there~~ exists a bounded pair (p^{o}, q^{o}) which is a topological potential in the wide sense for λ^{o} — ~~in fact,~~ a potential in the ordinary sense, since μ' and μ'' are also bounded. The strong Lindelöf property than implies that (p^{o}, q^{o}) is a *measure* potential for λ^{o}, from which the stated result follows (see ~~(18) of section 3~~ (3.18)). ///

We conclude our discussion of the transportation problem with a brief glance at the pioneering work of Kantorovitch.[15] (We ~~shall~~ use our notation and terminology to facilitate exposition.) He formulates a special case of the measure-theoretic transportation problem (variant I) — special in that the source and sink spaces are the same: $(A,\Sigma') = (B,\Sigma'')$, the cost function f is non-negative, μ' and μ'' are bounded, and a certain topological structure is imposed. Next, he defines a feasible flow λ as being *potential* iff ~~there~~ exists a (measurable) function $p{:}A \rightarrow$ reals such that

$$|p(b) - p(a)| \leq f(a,b)$$

display

break at 5

for all $a,b \in A$ (remember that $A = B$), and

$$p(b) - p(a) = f(a,b)$$

if (a,b) is a point of support for λ. His main assertion is ~~then~~ that a flow λ is optimal ~~if and only if~~ iff it is potential.

This is quite instructive, both in its accomplishments and its errors. The definition of p is close to our concept of "topological potential" for λ, differing from it in the minor point that the absolute value appears on the left of the inequality, and differing in the major point that just one function, p, appears, instead of a pair (p,q). ~~(In other words~~ That is, while the problem formulated is of the transportation form, the "potential" concept used is more appropriate for the transhipment problem considered below. This could not have happened if Kantorovitch did not identify the source and sink spaces). To place things into the transportation problem framework, think of these relations as defining conditions on the pair (p,~~q~~ p) rather than the single function p. Pair (p,p) is, indeed, a topological potential for λ if it satisfies these conditions. (p, p)

As for Kantorovitch's assertion, the "if" part is correct. We indicate how this can be demonstrated within the framework of our theory. The topological assumptions he makes imply the strong Lindelöf property, so that (p,p) is also a measure potential. The boundedness assumptions then guarantee optimality of λ.

The "only if" part is erroneous. If λ is an optimal flow one can indeed demonstrate the existence of a topological potential (p,q) under his assumptions (by our proof above). (Here both p and q will have domain A, since source and sink spaces are the same). But one cannot make the further assertion that p = q, as the following counterexample demonstrates. Let

$\underline{A} = \{a, b, c\}$, with all subsets open and measurable; $\mu'\{a\} = \mu''\{c\} = 1$; $\mu'\{b,c\} = \mu''\{a,b\} = 0$; $f(a,c) = 1$, $f = 0$ elsewhere. This satisfies all of Kantorovitch's premises. The optimal flow is: $\lambda^{o}\{(a,c)\} = 1$, and $\lambda^{o} = 0$ on all other singletons; in fact, this is the only feasible flow. Now suppose this flow were potential. We would have $p(c) - p(a) = 1$, since (a,c) is a point of support for λ^{o}. But also $p(c) - p(b) \leq 0$ and $p(b) - p(a) \leq 0$ — contradiction!

This three-page paper of Kantorovitch is the true <u>locus classicus</u> for the measure-theoretic transportation problem. The problem itself, the key rôle of "potentials", and certain basic solution methods, are all adumbrated here, even if the exposition is flawed. The energies of researchers have in the meantime been directed into other channels — (mainly to the development of "ordinary" programming methods), — so that the work of Kantorovitch appears to be the direct predecessor of ~~the present~~ this chapter, with a lag of some thirty-five years.

g. 23-6 (A)

7.6. Transhipment: Introduction

The transhipment problem with n locations is to find n^2 non-negative numbers x_{ij} $(i,j = 1,\ldots,n)$ satisfying

$$(x_{i1} + \ldots + x_{in}) - (x_{1i} + \ldots + x_{ni}) \leq \alpha_i \qquad \text{(7.6.1)}$$

$(i = 1,\ldots,n)$,

and minimizing the sum of

$$f_{ij}\ x_{ij} \qquad (7.6.2)\ \cancel{(2)}$$

over all n^2 terms of this form. Here the numbers α_i and f_{ij} ($i, j = 1,\ldots,n$) are given parameters.

The simplest interpretation is the following. x_{ij} is the quantity of a certain commodity moving from location i to location j. On the left of (1), the first parenthetical sum, $x_{i1} + \ldots + x_{in}$ (excluding x_{ii}), is the total quantity exported from location i to other locations; the second parenthetical sum (again excluding x_{ii}) is the total quantity imported to location i from other locations. Hence the left side of (1) may be thought of as net exports from location i. (The meaning of x_{ii} is problematical, but this term cancels out from the left side of (1) and therefore creates no interpretive difficulty). α_i may be thought of as the net capacity of location i. If positive, it gives the amount by which exports from i may exceed imports to i; if negative, it gives the amount by which imports to i must exceed exports from i. Constraint (1) states that net exports cannot exceed net capacity; we could just as well have stipulated that net imports cannot fall below net requirements β_i, where β_i is simply $-\alpha_i$.

~~Let us~~ Compare the constraint system (1) ~~above~~ with the constraints of the transportation problem, (1.1) & (1.2). ~~(1) and (2) of section 1.~~ We can think of the locations i for which α_i is positive as being sources, those for which α_i is negative as being sinks. (If $\alpha_i = 0$, i may be placed in either category).

The transportation problem allows flows only from sources to sinks, while the transhipment problem allows flows between any two sites, including source-to-source, sink-to-sink, and sink-to-source. Thus it becomes possible to "tranship" a flow from a source to a sink through a series of intermediate locations.

The objective function, (2) ~~above~~, has the same form as that for the transportation problem, (1.3) ~~(3) of section 1~~, though unit transport cost f_{ij} must now be defined for all pairs of locations, not just for source-sink pairs.

As with the transportation problem, we may distinguish variants of the transhipment problem. Thus (1)–(2) ~~above~~ will be called the inequality-constrained variant. The equality-constrained variant simply replaces "$\leq$" in (1) by "=".

The transhipment problem was first formulated by Orden, who also showed that there is a transportation problem that ~~which~~ is equivalent to the transhipment problem in a certain sense.[16] Indeed, several ways have been suggested for "reducing" transhipment to the transportation problem (or something resembling it). We ~~shall~~ explore one of these below.

First, ~~however,~~ it is worthwhile to compare the transportation and transhipment problems from the point of view of possible applications. We have ~~already~~ mentioned the interpretation of transhipment points as locations of physical Space. Specifically, imagine a system of cities, thought of as points, linked by a system of roads. A road directly links cities i and j iff it starts at i, ends at j, and passes through no other city. We then let f_{ij} be the transport cost incurred in moving a unit mass of the commodity from i to j over this direct link, or over the cheapest direct link if there are several. (If there is no direct link, set $f_{ij} = \infty$. Alternatively, we ~~one~~ may resort to the artifice of making f_{ij} finite but "very large", so that traffic avoids this "link" if at all possible).

Note ~~should be taken, by the way, of~~ the heroic linearity assumptions involved in the objective functions (2) ~~above~~ or (1.3) ~~(3) of section 1 —~~ or their generalizations to integrals. Congestion phenomena and scale economies (both very important in transportation) are ignored, and a doubling of traffic is assumed to double cost on any link. The concept of transport cost itself covers a motley collection of categories: fuel consumption, vehicle and road wear, travel time, risk of accident, discomfort, deterioration of cargo, traffic control costs, and perhaps vehicle and road construction costs, as well as pollution, noise, and other disamenities if all social costs are to be included.[17] 21

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Brushing aside all these conceptual difficulties, ~~then,~~ we postulate a unit cost f_{ij} associated with the cheapest direct link from location i to location j. ~~Let us~~ suppose f_{ij} is always finite, and let A be the set of all locations in the transhipment problem. The domain of f is then A × A, which is the same as that of a metric on A. Is it reasonable to assume that f *is* a metric? ~~Let us~~ examine the conditions one by one. Recall that $d:A \times A \to$ reals is a metric iff $d(a,a) = 0$, $d(a_1, a_2) > 0$ if $a_1 \neq a_2$, $d(a_1, a_2) = d(a_2, a_1)$, and $d(a_1, a_2) + d(a_2, a_3) \geq d(a_1, a_3)$.

As for the first property, $f_{ii} = 0$ has no clear empirical meaning; it does often turn out to be mathematically convenient to make this assumption. As for the second, while f_{ij} will usually be positive for $i \neq j$, it might be negative for some pairs (e.g. pleasure driving). Again, while f_{ij} and f_{ji} might be approximately equal, one can think of several reasons for inequality: going up-hill, up-stream, up-wind, vs. downhill, down-stream, down-wind; one-way streets; asymmetric bus routes; the ease of getting from little-known i to well-known j because of direction signs and road convergence.

This leaves the triangle inequality. Is it true that $f_{ij} + f_{jk} \geq f_{ik}$? Not necessarily — it may be less costly in going from i to k to tranship through j rather than take the direct link. It is almost obvious, in fact, that if the triangle inequality holds there is no rationale for transhipment: One does at least as well to ship directly from sources to sinks. (See section 10 below).

In summary, the unit transport cost function f need not be a metric. On the other hand, f's that satisfy some or all of the metric postulates ~~do~~ constitute interesting special cases. Even if f *is* a metric, however, it need not have any close resemblance to "real" geographic distance. The irregularities of nature, the construction of roads between some but not all places, irregular tariffs, and institutional barriers — all ~~these~~ conspire to weaken the relation between geographic distance and transport cost.

The points of the transhipment problem can also be interpreted as points of Time, or Space-Time, rather than points of ~~Space~~.[22] f_{ij} then becomes storage cost, or combined transport-storage cost. ~~f_{ij} would tend to be larger, the shorter the elapsed time-interval from i to j;~~ no ¶ (If j precedes i, we set $f_{ij} = \infty$, or to a number high enough to discourage traffic flow into the past.)

The relative advantages of the transhipment and transportation problem formulations may be summarized as follows. Because it allows connections between all pairs of points, the transhipment formulation allows the study of routing patterns and intermediate flows that ~~which~~ escape the transportation formulation. However ~~On the other hand~~, the very fact that all points are treated symmetrically, — rather than being dichotomized into sources and sinks, — means that a number of important interpretations of the transportation problem do not carry over to transhipment. In particular, this applies to the assignment of resources or land (as sources) to alternative activities (as sinks). The major application we make of the transportation problem ~~in fact~~ has this interpretation (see chapter 8); hence the latter is ~~of~~ much more important ~~importance~~ to us than transhipment ~~is~~.

Even on its own ground the transhipment formulation is not necessarily more useful than the transportation formulation. We have ~~already~~ mentioned that the former can be "reduced" to the latter in a number of ways. The following considerations — while not ~~, strictly speaking,~~ necessarily constituting a "reduction" — indicate another reason why the transportation formulation is often perfectly satisfactory (f is finite in the following discussion.) Dichotomize the points of a transhipment problem into "sources" and "sinks" according to the sign of the net capacity α_i. Suppose that for any pair consisting of a source i and a sink j there is a shortest route, $(k_1,\ldots,k_m)$, in the following sense: $k_1 = i$, $k_m = j$, and the sum

$$g_{ij} = f_{k_1k_2} + f_{k_2k_3} + \ldots + f_{k_{m-1}k_m} \qquad (7.6.3) \;\; \cancel{(3)}$$

is a minimum over all possible finite sequences of points, $(k_1,\ldots,k_m)$ satisfying $k_1 = i$, $k_m = j$. Choose one such route for each source-sink pair (i,j). It may then be verified that the transhipment problem, ~~(1)-(2) above~~ (1.1)–(1.3), "reduces" to the transportation problem ~~, (1)-(3) of section 1,~~ of all source-sink pairs, with capacities and requirements given by the absolute values $|\alpha_i|$, and unit transport costs g_{ij} given by (3) ~~above~~. "Reduction" here means that, if y^0_{ij} is an optimal flow for this transportation problem, an optimal flow for transhipment is obtained by shipping y^0_{ij} along each link of the shortest route from i to j, adding over all source-sink pairs (i,j).

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Now, if the shortest routes are easily found, or otherwise uninteresting, one might just as well go to the transportation problem derived above, which gives the optimal origin-destination flow pattern, and is easier to solve than the original.

To round out the discussion, we ~~briefly~~ mention the problem of finding a shortest route from i to j, one that minimizes (3). As pointed out by Orden, this can be formulated as a special case of the transhipment problem.~~[19]~~[23] ~~Namely,~~ viz. let $\alpha_i = +1$, $\alpha_j = -1$, $\alpha_k = 0$ for all other points. An optimal solution to this will yield one or more sequences $k_1, \ldots, k_m$, with $k_1 = i$, $k_m = j$ and positive flows between each successive pair. A little thought shows that each such sequence is a shortest route, and the minimal total cost for this problem is precisely g_{ij} of (3).

Does a shortest route always exist for any origin i, destination j? It does iff the following cyclic positivity condition is satisfied:

$$f_{k_1k_2} + \ldots + f_{k_{m-1}k_m} + f_{k_mk_1} \geq 0 \qquad (7.6.4) \quad ~~(4)~~$$

for all finite sequences $(k_1, \ldots, k_m)$, $m = 2,3,\ldots$. condition (4) states that the sum of unit costs around a closed circle of links is never negative. For suppose (4) were false for some sequence. By going around the circle sufficiently often one can drive cost below any negative number, hence there cannot be a minimum sum (3). Conversely, if cyclic positivity holds, there is no cost advantage to routes that ~~which~~ include the same point more than once. But there are just a finite number of routes without repeated points from i to j, hence a shortest route exists. (For the generalized transhipment problems discussed below, where the number of points may be infinite, the situation is much more complicated. But (4) remains a necessary condition for the existence of shortest routes).

Cyclic positivity is implied by, but weaker than, the triangle inequality. Indeed, with the triangle inequality, the pair (i,j) itself is a shortest route, and $g_{ij} = f_{ij}$ in (3 ~~4~~).

Finally, the measure-theoretic treatment of the transportation problem seems to be easier than that for transhipment.[24 ~~20~~]

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In the following sections ~~pages~~ we ~~shall~~ study the transhipment problem in measure-theoretic form. We ~~shall~~ emphasize ~~those~~ aspects in which transhipment is distinctive, where new and sometimes paradoxical phenomena appear. We ~~shall~~ also show how the use of pseudomeasures to formulate constraints arises naturally for transhipment. (Until now, ~~Up to this point in the book~~, pseudomeasures have been used only to represent preferences, with one small exception in Section 6.9).

A

7.7. Transhipment: Measure-Theoretic Formulations

We ~~shall~~ give two different measure-theoretic formulations of the transhipment problem. For the first, the raw materials are: a measurable space, $(\underline{A}, \Sigma)$; a ~~sigma~~ σ-finite signed measure, μ, on this space; and a function $f: \underline{A} \times \underline{A} \to$ reals, measurable with respect to the product ~~sigma~~ σ-field $\Sigma \times \Sigma$ on $\underline{A} \times \underline{A}$. The problem is to

Find a bounded measure λ on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$ satisfying

$$\lambda' - \lambda'' \leq \mu, \qquad (7.7.1)\ \cancel{(1)}$$

and minimizing

$$\int_{\Lambda} f \, d\lambda. \qquad (7.7.2)\ \cancel{(2)}$$

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Here λ', λ'' are the left and right marginals of λ, respectively, so that (1) could also be written in the following less abbreviated form:

$$\lambda(\underline{E} \times \underline{A}) - \lambda(\underline{A} \times \underline{E}) \leq \mu(\underline{E}), \qquad (7.7.3)\ \cancel{(3)}$$

~~for~~ all $\underline{E} \in \Sigma$. now (2) is an indefinite integral over $\underline{A} \times \underline{A}$, and "minimize" is to be understood in the sense of (reverse) standard ordering of pseudomeasures. This is the inequality-constrained variant; ~~one~~ we obtain~~s~~ the equality-constrained variant by substituting "=" for "≤" in (1) and (3). ~~above.~~

The signed measure μ is to be interpreted as net capacity, so that $\mu(\underline{E})$ is the amount by which the gross outflow from the points in set $\underline{E}$ may exceed the gross inflow to those points. $\mu(\underline{E})$ may, of course, be negative. λ is the flow measure, so that $\lambda(E \times F)$ gives the total mass ~~which~~ that moves (directly) from origins in E to destinations in F. In particular, $\lambda(E \times A)$ gives the gross outflow from origins in E to all destinations (including destinations in the set E); similarly, $\lambda(A \times E)$ gives the gross inflow to destinations in E from all origins. Thus (3) is precisely the relation between inflow, outflow, and capacity mentioned above. f is unit costs, and (2) gives the total cost of flow λ.

Care should be taken to distinguish ~~those~~ set functions ~~—~~ (such as λ) ~~— which are~~ defined on the product space $(A \times A, \Sigma \times \Sigma)$ from those ~~—~~ (such as λ', λ'', and μ) ~~— which are~~ defined on (A, Σ).

Problem (1)–(2) ~~above~~ reduce to ~~(1)-(2) of the preceding section~~ (6.1)–(6.2) precisely in the case when Σ is a finite ~~sigma~~ σ-field, so that we ~~do~~ indeed have a generalization of the original transhipment problem.

In ordinary transhipment ~~one~~ we distinguishes "source" points from "sink" points by the sign of the net capacity. In the generalization (1) this role is played by the Hahn decomposition of net capacity μ. ~~Indeed,~~ If (P,N) is a Hahn decomposition, ~~the~~ then $\mu(E) \geq 0$ for all measurable $E \subseteq P$, $\mu(F) \leq 0$ for all measurable $F \subseteq N$, so that the points of P may be thought of as "sources", the points of N as "sinks".

~~Note that~~ Boundedness is a feasibility condition for λ. Indeed, if λ were unbounded, ~~then~~ (1) would not be well-defined, since we would have $\lambda'(A) = \lambda''(A) = \infty$. ~~On the other hand,~~ However, boundedness ~~this constraint~~ excludes some interesting theoretical situations. On the endless plane of location theory there will usually be a flow of infinite mass. The same is true with an unbounded time-horizon, in those cases where A is a subset of Time, or Space-Time. In these cases a similar question arises concerning the adequacy of a signed measure to represent the concept of "net capacity". Suppose ~~that~~ A is split as above into two pieces, a "source" space P, and a "sink" space N. Since μ is a signed measure, at least one of the two numbers, $\mu(P)$, $\mu(N)$, must be finite. But there are reasonable problems involving both infinite capacity on P and infinite requirements on N.

Our second measure-theoretic formulation enables us to deal with the situations just discussed. As might have been expected, the key lies in the introduction of pseudomeasures. (A,Σ) and f remain as above. The objective is still to minimize (2), but (1) and the boundedness condition are replaced. Instead of the signed measure μ we have a pseudomeasure ψ on (A,Σ), and the constraint is: to

Find a measure λ on $(A \times A, \Sigma \times \Sigma)$ whose marginals, λ' and λ'', are ~~sigma~~ σ-finite and satisfy:

$$(\lambda', \lambda'') \leq \psi. \qquad \text{(7.7.4)} \; ~~(4)~~$$

That is, we form the pseudomeasure (λ', λ'') from the marginals of λ, and constrain ~~it~~ i to be less than or equal to

ψ under narrow ordering. Letting (ψ^+, ψ^-) be the Jordan form of ψ, (4) may be written in less abbreviated form as ~~follows:~~

$$\lambda(E \times A) + \psi^-(E) \leq \lambda(A \times E) + \psi^+(E), \qquad \text{(7.7.5)}\ \cancel{(5)}$$

for all $E \in \Sigma$. This is the unbounded formulation of the transhipment problem, (1) giving the bounded formulation.

When λ is bounded, and ψ is a signed measure μ, ~~one~~ we can ~~easily~~ see~~s~~ that (5) is the same as (3). Thus the constraint (1) is a special case of (4). But it cannot be said that the bounded formulation is merely a special case of the unbounded one, since the boundedness condition is present in one and absent in the other. As above there is also an equality-constrained variant: just substitute "=" for "$\leq$" in (4) and (5).

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The pseudomeasure (λ', λ'') may be thought of as net outflow or net exports, and the fact that it is a pseudomeasure allows the possibility that gross inflow and outflow for a region may both be infinite. ψ is again net capacity. If (P,N) is a Hahn decomposition for ψ, we may think of it in the following way: ψ^+ gives the net outflow capacity on source space P, while ψ^- gives the net inflow requirement on sink space N.

The conditions that λ' and λ'' are ~~sigma~~ σ-finite are needed to make (λ', λ'') well-defined as a pseudomeasure. Recall that either of these implies that λ itself is ~~sigma~~ σ-finite, so that (2) remains well-defined as a pseudomeasure. ~~(Note that~~ preference among different λ's is expressed via (2) by standard ordering, while the constraint (4) involves narrow ordering. This disparity is essential to achieve a bona fide generalization of the ordinary transhipment problem.)

We have now set up the two measure-theoretic transhipment problems, and ~~shall~~ will investigate feasibility and duality conditions for them. But first we ~~shall finish this introductory section by~~ deriving some results concerning measures λ on a product space of the form $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$. The aim is to achieve a certain insight into the structural differences between the transportation and transhipment constraints. The following remarks are abstracted from any particular problem context, however. ~~Note also that~~ they also apply to arbitrary measures λ, not merely to sigma-finite measures.

Definition: Measure λ on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$ is a translocation[21] iff there is a measurable partition, $\{\underline{P},\underline{N}\}$, of $\underline{A}$ into two pieces such that $\lambda\left[(\underline{A} \times \underline{A})\setminus(\underline{P} \times \underline{N})\right] = 0$.

(That is, $\lambda(\underline{P} \times \underline{P}) = \lambda(\underline{N} \times \underline{N}) = \lambda(\underline{N} \times \underline{P}) = 0$. The only possible flow is from $\underline{P}$ to $\underline{N}$.)

Theorem: λ is a translocation iff its marginals, λ' and λ", are mutually singular.

Proof: Let λ be a translocation, with $\underline{P}$, $\underline{N}$ as in the definition above. Then $\lambda'(\underline{N}) = \lambda''(\underline{P}) = 0$, so λ', λ" are mutually singular.

Conversely, let λ', λ" be mutually singular, so that $\lambda'(\underline{N}) = \lambda''(\underline{P}) = 0$ for some partition $\{\underline{P},\underline{N}\}$ of $\underline{A}$. But this yields $\lambda(P \times P) = \lambda(N \times N) = \lambda(N \times P) = 0$, so λ is a translocation.

~~Next,~~ we want a formula giving the transhipment associated with any flow measure λ. Intuitively, the transhipment in a region is given by the "overlap" between inflow and outflow. To be precise, let transhipment be represented by a measure θ on (A,Σ). We require that $\theta \leq \lambda'$ and $\theta \leq \lambda''$ ~~— that is,~~ i.e., transhipment in any region does not exceed gross outflow from that region, and does not exceed gross inflow into that region, respectively. The "overlap" is the largest measure meeting these conditions. But this is precisely the infimum of λ' and λ'', as defined in ~~chapter 3,~~ section 3. 1. Thus we have the following definition.

Definition: Given measure λ on $(A \times A, \Sigma \times \Sigma)$, the transhipment is the measure θ on (A,Σ) given by

$$\theta(E) = \inf(\lambda',\lambda'')(E) = \inf\left\{\lambda'(F) + \lambda''(E \setminus F) \,\middle|\, F \subseteq E,\ F \in \Sigma\right\}, \qquad (7.7.6)$$

for all $E \in \Sigma$.

λ prime

Here λ', λ'' are the left and right marginals of λ, respectively. We have repeated the explicit formula for $\inf(\lambda',\lambda'')$ for convenience.

This definition seems to capture quite well the intuitive notion of "transhipment". In particular, consider the case when λ is a translocation. Here the marginals are mutually singular; there is no overlap, and transhipment should be zero.

Furthermore, the converse should be true. If transhipment is zero, ~~then~~ inflow and outflow should be mutually singular, so that λ is a translocation. The following result confirms this expectation. Note that we are actually proving an abstract theorem: inf $(\mu,\nu) = 0$ iff (μ,ν) is a mutually singular pair.

D

Theorem: λ has a zero transhipment iff λ is a translocation.

Proof: Let λ be a translocation, so that $\lambda'(N) = \lambda''(P) = 0$ for some partition $\{P,N\}$ of A, by the theorem above. Then

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$$\theta(A) \leq \lambda'(N) + \lambda''(P) = 0,$$

from (6) with $F = N$. Hence $\theta = 0$.

Conversely, let $\theta(A) = 0$. Then, for each $n = 1, 2, \ldots,$ ~~there is~~ a set $F_n \in \Sigma$ exists such that

$$\lambda'(F_n) + \lambda''(A \setminus F_n) \leq 2^{-n},$$

from (6). Let $F = \limsup F_n$. For each $n = 1, 2, \ldots$ we have

$$\lambda'(F) \leq \lambda'(F_n \cup F_{n+1} \cup \ldots) \leq 2^{-n} + 2^{-(n+1)} + \ldots = 2 \cdot 2^{-n}. \quad 2 \cdot 2^{-n}.$$

Hence $\lambda'(F) = 0$. Also $A \setminus F = \liminf(A \setminus F_n)$, so that $\lambda''(A \setminus F)$ does not exceed the sum of

$$\lambda''\left[(A \setminus F_n) \cap (A \setminus F_{n+1}) \ \ldots\right] \qquad \text{(7.7.7)} \ \cancel{(7)}$$

over $n = 1,2,\ldots$. But each term (7) equals zero, since it does not exceed $\lambda''(A \setminus F_k)$ for arbitrarily high k. Hence

D $\lambda''(\underline{A}\backslash\underline{F}) = 0$. This with $\lambda'(\underline{F}) = 0$ shows that λ', λ'' are mutually singular, hence λ is a translocation, by the preceding theorem. □

Measures λ', λ'' have been interpreted as the gross outflow and inflow, respectively, associated with λ. There are a number of intuitive net flow concepts; particularly ~~The one we have in mind here is~~ that ~~which~~ nets out the "overlap" of λ' and λ'' from each of them, i.e. ~~— that is~~, subtracts the transhipment. ~~The trouble is that~~ However, these measures may all be infinite, so ~~that~~ subtraction is not a well-defined operation.

But recall that, in ~~chapter 3,~~ section 3.1, we did define a subtraction operation that ~~which~~ is valid for infinite measures. The concepts in that section, in fact, turn out to be admirably well suited to explicate the intuitive notions we are struggling with here.

D **Definition**: Let λ be a measure on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$. The net outflow and net inflow of λ are the respective measures λ_1 and λ_2 on $(\underline{A},\Sigma)$ given by

breakout:

$$(\lambda_1, \lambda_2) = \underline{J}(\lambda', \lambda'').$$

That is, λ_1 and λ_2 are the upper and lower variations, respectively, of the Jordan decomposition of the pair (λ',λ'').

That this is a reasonable definition follows from the basic relation between pairs of measures, their Jordan decompositions, and their infima, which in this case is

$$\lambda_1 = \lambda' - \inf(\lambda', \lambda'') = \lambda' - \theta,$$

$$\lambda_2 = \lambda'' - \inf(\lambda', \lambda'') = \lambda'' - \theta.$$

(See ~~page~~ ... .) That is, the net outflows and inflows are indeed obtained by subtracting transhipment from gross outflows and inflows, respectively. (If θ is finite, this reduces to ordinary subtraction).

Another intuitively appealing property ~~that~~ one would wish the net flow measures λ_1 and λ_2 to possess is that they be <u>mutually</u> <u>singular</u>. For in this case one can split $\underline{A}$ into two pieces, $\underline{P}$ and $\underline{N}$, which can be unambiguously labeled as the outflow and inflow sets, respectively. (Here $\lambda_1(\underline{N}) = \lambda_2(\underline{P}) = 0$.) An obvious sufficient condition for this is that λ be a translocation; for then even the gross flows, λ' and λ'', are mutually singular, hence <u>a</u> <u>fortiori</u> the net flows, λ_1 and λ_2. (~~In fact~~ $\lambda_1 = \lambda'$ and $\lambda_2 = \lambda''$ in this case, since $\theta = 0$). The following result shows that, even if λ is not a translocation, mutual singularity is guaranteed under quite general conditions.

<u>Theorem</u>: Let measure λ on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$ be <u>abcont</u>. Then its net flows, λ_1 and λ_2, are mutually singular.

<u>Proof</u>: Marginal λ' is induced from λ by the projection $(\underline{a}_1, \underline{a}_2) \mapsto \underline{a}_1$. Since λ is abcont, so is λ'. Hence the pair (λ', λ'') is Hahn decomposable, implying that its Jordan decomposition, (λ_1, λ_2), is a mutually singular pair (See Section 3.1.). |||

Finally, ~~let us~~ tie these concepts to the transportation-transhipment problem dichotomy. We show ~~in fact~~ that the transportation problem (variant I) is essentially the transhipment problem (equality-constrained unbounded formulation) with an extra constraint thrown in, that λ be a translocation.

23-12 Start with the transhipment problem whose feasible set is determined by the pseudomeasure space $(\underline{A},\Sigma,\psi)$. Measure λ is feasible iff the marginals λ', λ'' are ~~sigma~~ σ-finite and

(7.7.8)

$$(\lambda',\lambda'') = \psi. \tag{8}$$

Then (8) is an equality between pseudomeasures (cf. (4)). Now add the additional constraint that λ must be a translocation. It follows that λ', λ'' are mutually singular, so that (λ',λ'') is ~~in fact~~ the Jordan form of ψ: $\lambda' = \psi^+$ and $\lambda'' = \psi^-$. Let $\{\underline{P},\underline{N}\}$ be a partition of $\underline{A}$ such that $\psi^+(\underline{N}) = \psi^-(\underline{P}) = 0$. Then λ is zero when restricted to $(\underline{A} \times \underline{A})\setminus(\underline{P} \times \underline{N})$. Let λ_o be λ restricted to $\underline{P} \times \underline{N}$, let μ' be ψ^+ restricted to $\underline{P}$, and let μ'' be ψ^- restricted to $\underline{N}$; also let Σ', Σ'' be Σ restricted to $\underline{P}$, $\underline{N}$, respectively. Then ~~it is easy to~~ we can see that λ_o is feasible for the (variant I) transportation problem, with source and sink spaces $(\underline{P},\Sigma',\mu')$, $(\underline{N},\Sigma'',\mu'')$, respectively.

CONT. gal. 23-12 550 LEAD TO go!

Conversely, given this transportation problem with feasible flow λ_o, this entire procedure may be reversed to yield a translocation λ satisfying (8). Furthermore, if $\underline{f}:\underline{A} \times \underline{A} \to$ reals determines the transhipment objective function, and $\underline{f}_o$ is $\underline{f}$

restricted to $\underline{P} \times \underline{N}$, then $\int \underline{f}_o \, d\lambda_o$ yields the same ordering among feasible transport flows $\underline{\lambda}_o$ as $\int \underline{f} \, d\lambda$ does among the corresponding translocations λ. This shows the essential equivalence between these two problems.

A

7.8. Transhipment: Feasibility

We now investigate the conditions under which feasible solutions exist for the bounded and unbounded formulations of the transhipment problem. The bounded case is well-behaved, and the results are analogous to those obtained for the transportation problem. But the results for the unbounded case are "wild".

First we discuss ~~for~~ the bounded formulation, (7.1) ~~(1) of section 7~~. Actually, we ~~shall~~ prove results for a somewhat more general problem: ~~We shall~~ let net capacity be a pseudomeasure ψ, and not ~~restrict attention~~ merely ~~to~~ a σ ~~sigma~~-finite signed net capacity measure~~s~~ μ. This yields a problem somewhere in between the bounded and unbounded formulations~~; namely:~~

Find a bounded measure λ on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$ satisfying

$$(\lambda', \lambda'') \leq \psi. \qquad (7.8.1) \; \cancel{(1)}$$

Here the marginals λ', λ'', as well as the pseudomeasure ψ, are ~~all~~ defined on the space $(\underline{A}, \Sigma)$ as usual. The left side of (1) could also be written as $\lambda' - \lambda''$, as in (7.1) ~~(1) of section 7~~, but we prefer the pseudomeasure notation.

The reader may wonder why we used constraint (7.1) ~~(1) of section 7~~ instead of the more general (1) ~~above~~. The answer is contained in the following theorem. If ψ is a proper pseudomeasure ~~—~~ (i.e. ~~that is~~, if ψ^+ and ψ^- ~~are both infinite measures~~) ~~—~~ then there is no bounded λ satisfying (1) ~~above~~. Hence pseudomeasures are actually useless here, and we ~~one~~ might just as well use the signed measure formulation of (7.1) ~~(1) of section 7~~, which is after all much closer to intuition than (1) ~~above~~ is. But we ~~one~~ need~~s~~ to formulate the problem (1) ~~above~~ to prove this ~~very~~ fact.

#

¶ **Theorem:** Given pseudomeasure space $(\underline{A},\Sigma,\psi)$, ~~there~~ exists a bounded measure λ on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$ satisfying (1) iff

D

$$\psi^+(\underline{A}) \geq \psi^-(\underline{A}) < \infty. \qquad (7.8.2)\ \cancel{(2)}$$

Also, ~~there~~ exists a bounded λ satisfying (1) with equality iff

$$\psi^+(\underline{A}) = \psi^-(\underline{A}) < \infty. \qquad (7.8.3)\ \cancel{(3)}$$

#

¶43 ¶ **Proof:** Let bounded λ satisfy (1), which may also be written:

$$\lambda' + \psi^- \leq \lambda'' + \psi^+. \qquad (7.8.4)\ \cancel{(4)}$$

D

We have $\lambda'(\underline{A}) = \lambda(\underline{A} \times \underline{A}) = \lambda''(\underline{A}) < \infty$. Hence, substituting $\underline{A}$ into (4), the λ terms drop out and we have $\psi^-(\underline{A}) \leq \psi^+(\underline{A})$. Next, let $(\underline{P},\underline{N})$ be a Hahn decomposition for ψ. Then $\psi^-(\underline{P}) = 0$, and $\psi^-(\underline{N}) \leq \lambda''(\underline{N})$ results from substituting $\underline{N}$ into (4). Hence

$\psi^-(\underline{A})$ is finite. This yields (2).

If bounded λ satisfies (1), hence (4), <u>with equality</u>, the same argument yields (3).

Conversely, let (2) obtain, and consider the <u>transportation</u> problem with source space $(\underline{A},\Sigma,\psi^+)$, ~~and~~ sink space $(\underline{A},\Sigma,\psi^-)$, and constraints

$$\lambda' \le \psi^+,\ \lambda'' = \psi^-. \qquad (7.8.5)\ \cancel{(5)}$$

This is variant II, hence a feasible solution λ exists, by (2). Constraints (5) impl~~ies~~y (4), which is (1). Also $\lambda''(\underline{A}) = \psi^-(\underline{A})$, hence λ is bounded, again by (2). Thus λ is feasible for the bounded transhipment problem.

Finally, let (3) obtain, and consider the same transportation problem, except that (5) has all equalities. This is variant I, hence a solution exists by (3). (5) now yields (1) with equality. λ is again bounded, since $\psi^-(\underline{A})$ is finite. |||

When ψ is the signed measure μ, as in (7.1) ~~(1) of section 7~~, this theorem takes on a very simple form.

<u>Theorem</u>: ~~There exists~~ a bounded measure satisfying (7.1) ~~(1) of section 7~~ iff $\mu(\underline{A}) \ge 0$. ~~There exists~~ a bounded measure satisfying (7.1) ~~(1) of section 7~~ <u>with equality</u> iff $\mu(\underline{A}) = 0$.

<u>Proof</u>: Proof is Immediate from (2)–(3) ~~above~~, noting that $\mu(\underline{A}) = \mu^+(\underline{A}) - \mu^-(\underline{A})$. |||

Thus a solution exists iff total net capacity is non-negative or zero, in the inequality- or equality-constrained problems, respectively. ~~As mentioned above,~~ this bears comparison with the transportation problem result. Here a feasible solution exists iff total requirement does not exceed total capacity (in variants II, III, IV, which involve inequality constraints), or iff total requirement equals total capacity (in the all-equality-constraint variant I).

This brings us to the unbounded formulation of transhipment, (7.4) ~~(4) of section 7~~. The basic result is that ~~there~~ always exists a feasible solution (unless Σ is finite). This is highly paradoxical, since a solution exists even when $\psi^{+}(\underline{A})$ is less than $\psi^{-}(\underline{A})$ -- (even when the former is zero and the latter infinite, in fact). We ~~shall~~ first prove the result and then give a rough explanation of "why" it is true. In the following we prove feasibility for the equality-constrained problem. The solution constructed automatically remains feasible for the weaker inequality constraint, so that feasibility holds in general.

Theorem: Let $(\underline{A},\Sigma,\psi)$ be a pseudomeasure space, with Σ an infinite σ~~sigma~~-field. Then ~~there~~ exists a measure λ on $(\underline{A} \times \underline{A},\ \Sigma \times \Sigma)$ such that the marginals λ', λ'' are σ~~sigma~~-finite, and

$$(\lambda',\lambda'') = \psi. \qquad (7.8.6)\ ~~(6)~~$$

(Equality in the sense of pseudomeasures).

(1.4)

Proof: Choose a representative, (μ,ν), of ψ. Since Σ is infinite and μ, ν are ~~sigma~~ σ-finite, ~~there exists~~ an infinite countable measurable partition, $\{A_1, A_2,\ldots\}$ of A into nonempty sets exists, such that $\mu(A_n)$ and $\nu(A_n)$ are finite for all $n = 1, 2,\ldots$.

Choose a point $a_n \in A_n$ for each n, and define the set function λ, with domain $\Sigma \times \Sigma$, as follows. For each $G \in \Sigma \times \Sigma$, and each $n = 1, 2,\ldots$, form the quantity

$$\mu\{a \mid a \in A_n \text{ and } (a,a_n) \in G\} + \nu\{a \mid a \in A_n \text{ and } (a_n,a) \in G\} + \mu_n \cdot x_n(G) + \nu_n \cdot y_n(G). \tag{7.8.7}$$

~~(7)~~

Here μ_n and ν_n are abbreviations for $\mu(A_n)$, $\nu(A_n)$ respectively; $x_n(G)$ is the number of integers $k \geq n$ for which $(a_k, a_{k+1}) \in G$; $y_n(G)$ is the number of integers $k \geq n$ for which $(a_{k+1}, a_k) \in G$. (If the number of such integers is infinite, take $x_n(G)$ or $y_n(G)$ to be $+\infty$, and form (7) by the rules of extended real-valued arithmetic). Finally, $\lambda(G)$ is defined as the sum of the quantities (7) over all $n = 1, 2,\ldots$.

We claim that λ constructed in this way is the desired measure. First of all, for fixed n each of the terms in (7) is routinely verified to be a measure on $\Sigma \times \Sigma$; hence λ, as a sum of measures, is itself a measure.

Now consider various sets $E \in \Sigma$ in relation to the points $a_1, a_2,\ldots$. First, if none of these points belongs to E, we calculate from (7) that

$$\lambda(E \times A) = \mu(E), \quad \lambda(A \times E) = \nu(E). \qquad \text{(7.8.8)}\ \cancel{(8)}$$

Second, if none of these points belongs to E, with the single exception of a_m, ~~then~~ a more complicated calculation from (7) yields

$$\lambda(E \times A) = \mu(E) + \mu_1 + \ldots + \mu_m + \nu_1 + \ldots + \nu_m, \qquad \text{(7.8.9)}\ \cancel{(9)}$$

and

$$\lambda(A \times E) = \nu(E) + \mu_1 + \ldots + \mu_m + \nu_1 + \ldots + \nu_m. \qquad \text{(7.8.10)}\ \cancel{(10)}$$

In particular, $E = A_m$ has the property just mentioned. Hence $\lambda'(A_m) = \lambda(A_m \times A)$ is finite for all $m = 1, 2, \ldots$, from (9). Similarly, $\lambda''(A_m)$ is finite for all m, from (10). Hence the marginals λ', λ'' are σ ~~sigma~~-finite.

Furthermore, from (8)–(10) ~~(8), (9), and (10)~~ we find that

$$\lambda'(E) + \nu(E) = \lambda''(E) + \mu(E) \qquad \text{(7.8.11)}\ \cancel{(11)}$$

for any set $E \in \Sigma$ to which at most one of the points $a_1, a_2, \ldots$ belongs. (E) But, However, ~~on the other hand,~~ any set $E \in \Sigma$ can be countably partitioned into sets of this type: $E = (E \cap A_1) \cup (E \cap A_2) \cup \ldots$. Hence, by summation, (11) is true for all $E \in \Sigma$. But this implies (6), by the equivalence theorem for pseudomeasures. Hence λ is feasible. ||| □

Measure λ "works" in the foregoing proof for the following reasons. ~~First of all,~~ the point a_n functions as a "depot" or "entrepôt" between A_n and the rest of A. a_1 absorbs any surplus

(5)

or deficit arising in A_1; a_2 does the same for A_2, and also absorbs the net surplus or deficit at a_1; a_3 does the same for A_3, and also absorbs the cumulative net surplus or deficit at a_2, etc. In this way, each successive set A_n is brought into balance, while the overall surplus or deficit "escapes to – or from – infinity". The paradox arises precisely because there is no point at which the buck stops and accounts must be settled. Similar phenomena arise in other contexts – e.g. ~~for example~~, in the theory of Markov chains with an infinite number of states, or in the theory of economic growth with intergenerational transfers and an infinite succession of generations.

One might be tempted to regard this paradox as a *reductio ad absurdum* of the unbounded formulation of the transhipment problem; but this would be an error, or at least a premature judgment. The formulation itself arises in a natural way. And even though a paradoxical flow pattern is *feasible*, it involves a great deal of cross-hauling. We may presume, then, that no such flow would be *optimal*, unless the problem is formulated in a way that allows no avoidance of such flows (by making requirements exceed capacities). It is quite common for useful models to introduce artifacts of this sort. Finally, many former "paradoxes" are now accepted as valid, so one should be wary of making summary judgments about what cannot occur in the real world.

The premise that Σ is an infinite σ ~~sigma~~-field is essential in this theorem. Indeed, if Σ is finite, we are in effect back in the ordinary transhipment problem, and the "tame" feasibility results of the bounded formulation apply.

7.9. Transhipment: Duality

We ~~shall~~ give a brief introduction to the duality theory of the transhipment problem, one ~~which~~ that parallels the treatment of the transportation problem. Returning ~~first~~ to the ordinary transhipment problem, ~~(1)-(2) of section 6~~ (6.1) and (6.2), its linear programming dual is the following:

Find non-negative numbers, $p_1, \ldots, p_n$ satisfying

$$p_j - p_i \leq f_{ij}$$

$(i, j = 1, \ldots, n)$, and maximizing

$$-\alpha_1 p_1 - \ldots - \alpha_n p_n. \qquad \text{~~(1)~~ (7.9.1)}$$

(Minus signs appear in the objective function (1) because we expressed the primal in terms of net capacities, α_i $(i = 1, \ldots, n)$. If we had used net requirements instead, we would get plus signs.)

The dual of the corresponding measure-theoretic problem (unbounded formulation), ~~(4) and (2) of section 7~~ (7.4) and (7.2), is defined as follows:

Find a measurable non-negative function $p: A \rightarrow$ reals satisfying

$$p(a'') - p(a') \leq f(a', a'') \qquad \text{~~(2)~~ (7.9.2)}$$

for all a', $a'' \in A$, and maximizing

$$-\int p \, d\psi. \qquad \text{~~(3)~~ (7.9.3)}$$

Here (3) is an indefinite integral over space ($\underline{A}$,Σ), and "maximization" is to be understood in the sense of standard ordering.

This is for the inequality-constrained version. For the equality-constrained version the dual is the same, except that p need not be non-negative. Finally, for the bounded formulation, everything is as above except for notation: The signed measure μ replaces the more general pseudomeasure ψ in (3).

The dual for the transportation problem introduced some other constraints making certain definite integrals well-defined and finite. Conditions of this sort play a rôle here, too, but it is convenient to introduce them separately.

The following theorem yields the basic duality inequality. It applies to both equality- and inequality-constrained versions, and to both bounded and unbounded formulations. The notation for the latter will be used (for the former, replace ψ by μ). The expression $\int_{\underline{A}} p\, d\psi$ means the following: It is defined iff the two definite integrals, $\int_{\underline{A}} p\, d\psi^+$ and $\int_{\underline{A}} p\, d\psi^-$, are both well-defined and not infinite of the same sign. In this case we set

$$\int_{\underline{A}} p\, d\psi = \int_{\underline{A}} p\, d\psi^+ - \int_{\underline{A}} p\, d\psi^- . \qquad \text{~~(4)~~ (7.9.4)}$$

(Equivalently, the expression is defined iff $\int p\, d\psi$ is a signed measure, and in this case $\int_{\underline{A}} p\, d\psi$ is its value at $\underline{A}$.)

Theorem: Let measure λ on space $(A \times A, \Sigma \times \Sigma)$ be feasible for the transhipment problem, and let $p{:}A \to$ reals be feasible for the corresponding dual problem. Also assume that

$$\int_A p \, d\lambda', \int_A p \, d\lambda'' \qquad (7.9.5) \quad \cancel{(5)}$$

are both well-defined and finite definite integrals. Then the following two definite integrals are well-defined, and the stated inequality holds between them:

$$\int_{A\times A} f \, d\lambda \geq -\int_A p \, d\psi. \qquad (7.9.6) \quad \cancel{(6)}$$

Proof: Let the functions $p', p''{:}A \times A \to$ reals be given by:

$$p'(a',a'') = p(a'), \quad p''(a',a'') = p(a''),$$

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for all $a', a'' \in A$. Condition (2) then takes the form: $p'' - p' \leq f$, and we obtain

one line if possible

$$\int_A p \, d\lambda'' - \int_A p \, d\lambda' = \int_{A\times A} p'' d\lambda - \int_{A\times A} p' \, d\lambda \qquad (7.9.7) \quad (7)$$

$$= \int_{A\times A} (p''-p') d\lambda \leq \int_{A\times A} f \, d\lambda.$$

The first equality in (7) arises from the induced integrals theorem. Note that

$$\int_{A\times A} (p'' - p')^{-}\, d\lambda \geq \int_{A\times A} f^{-}\, d\lambda, \qquad (7.9.8)\ \cancel{(8)}$$

from (2). The left side of (8) is finite, from (5), hence so is the right, hence the last integral in (7) is well-defined. The inequality in (7) then follows from (2).

Next, we prove that

$$\int_A p\, d\lambda'' - \int_A p\, d\lambda' \geq \int_A p\, d\psi^{-} - \int_A p\, d\psi^{+}, \qquad (7.9.9)\ \cancel{(9)}$$

the differences being well-defined. There are two cases.

For the inequality-constrained variant, we have $p \geq 0$ and $(\lambda', \lambda'') \leq \psi$; i.e. ~~– that is~~,

$$\lambda' + \psi^{-} \leq \lambda'' + \psi^{+}, \qquad (7.9.10)\ \cancel{(10)}$$

so that

$$\int_A p\, d\lambda' + \int_A p\, d\psi^{-} \leq \int_A p\, d\lambda'' + \int_A p\, d\psi^{+}. \qquad (7.9.11)\ \cancel{(11)}$$

Letting (P, N) be a Hahn decomposition for ψ, we have $\psi^{-}(P) = 0$, while

$$\psi^{-}(E) \leq \lambda'(E) + \psi^{-}(E) \leq \lambda''(E) + \psi^{+}(E) = \lambda''(E)$$

for any measurable $E \subseteq N$. Hence $\psi^{-} \leq \lambda''$. It follows from (5) that at least three of the integrals in (11) are finite. Hence it is permissible to rearrange terms to obtain (9).

For the equality-constrained variant, (10) holds with equality. Also, by the minimizing property of the Jordan form, we have $\psi^+ \leq \lambda'$, as well as $\psi^- \leq \lambda''$. It then follows from (5) that all the integrals appearing in (11) are finite. Hence (11) holds (with equality) and may be rearranged to yield (9) (with equality).

(7), (9), and (4) together yield (6). ///

Next, we look for a condition under which the inequality (6) of this theorem becomes an equality.

Definition: Let measure λ on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$, and the function $\underline{p}:\underline{A} \rightarrow$ reals, be feasible for the transhipment problem and its dual, respectively. Function $\underline{p}$ is a (transhipment) measure potential for λ iff the following two conditions are satisfied:

$$\lambda\left\{(a_1,a_2) \middle| \underline{p}(a_2) - \underline{p}(a_1) < \underline{f}(a_1,a_2)\right\} = 0, \tag{12}$$

and, when restricted to the subset $\{a|\underline{p}(a) > 0\}$ of $\underline{A}$, the two pseudomeasures, (λ',λ'') and ψ, coincide.

This definition is meant to apply to both bounded and unbounded formulations, and to both equality- and inequality-constrained variants, of the transhipment problem. Note, however, that for the equality-constrained variant, the second condition is satisfied trivially and may be dropped; measure-potentiality reduces to (12) alone. Condition (12) states that there is no flow on the set of origin-destination pairs for which the dual inequality (2) is strict. This and the other measure-potentiality condition generalize the complementary slackness conditions for transhipment.

Theorem: Let measure λ° on $(A \times A, \Sigma \times \Sigma)$ be feasible for the transhipment problem, and $p^\circ : A \rightarrow$ reals be feasible for its dual. Also let

$$\int_A p^\circ \, d\lambda^{\circ\prime}, \quad \int_A p^\circ \, d\lambda^{\circ\prime\prime}$$

both be well-defined and finite. Then p° is a (transhipment) measure potential for λ° iff

$$\int_{A \times A} f \, d\lambda^\circ = -\int_A p^\circ \, d\psi. \qquad (13)$$

Proof: Let p° be a measure potential for λ°. Reviewing the preceding proof, we find that the weak inequality in (7) is satisfied with equality because of (12). For the equality-constrained variant this already yields (13), since (9) is also satisfied with equality. For the inequality-constrained variant the fact that $(\lambda', \lambda'') = \psi$ on the set $\{a \mid p(a) > 0\}$ yields (10) with equality on this set. Hence (11) and (9) are satisfied with equality, since $p = 0$ off this set; this again yields (13).

Conversely, assume (13). All integrals in the preceding proof are then finite, and the weak inequalities in (7) and

(9) are satisfied with equality. But equality in (7) implies (12), while equality in (9) implies that $(\lambda', \lambda'') = \psi$ when these are restricted to $\{a | p(a) > 0\}$ (a trivial implication in the equality-constrained variant.) □

Theorem: Let λ^o be a **bounded** measure on space $(A \times A, \Sigma \times \Sigma)$, and let $p^o : A \to$ reals be a **bounded** function, such that p^o is a (transhipment) measure potential for λ^o. Then λ^o is best for the **bounded** formulation of the transhipment problem.

Proof: Let λ be any other feasible solution for the transhipment problem (bounded formulation). We show that

$$\int_{A \times A} f \, d\lambda^o = -\int_A p^o \, d\mu \leq \int_{A \times A} f \, d\lambda, \qquad (7.9.14) \quad (14)$$

all these definite integrals being well-defined. (Here f is, as usual ~~of course,~~ the unit cost function, and μ is the net capacity signed measure.) First,

$$\int_A p^o \, d\lambda', \int_A p^o \, d\lambda'' \qquad (7.9.15) \quad (15)$$

are both well-defined and finite, since p^o and λ are both bounded. This yields the inequality in (14), by (6). ~~(15)~~ (15) remains finite if λ is replaced by λ^o, and this, ~~together~~ with the measure-potentiality premise, yields the equality in (14), by (13). Furthermore, this common value is finite. (14) then implies that $\lambda^{\underline{o}}$ is best under (reverse) standard ordering of pseudomeasures. |||

These results apply to both the equality- and inequality-constrained variants. Note that the last theorem applies only to the bounded formulation of the transhipment problem, however. In the unbounded formulation, the integrals (15) will not necessarily be well-defined and finite for all feasible λ, which means that the inequality of (14) cannot be derived.

In connection with transhipment potentials one should mention the work of Martin Beckmann.[~~22~~ 26] This deals with commodity flow on the plane, and makes essential use of vector analysis (gradients, curls, etc.). Here "flow" refers to "continuous" physical movement (as a fluid) and is not immediately reducible to the origin-destination form of the transhipment problem. Yet Beckmann ~~he~~ arrives at a potential function ~~which is~~ similar to the transhipment potential. One hopes that future work will produce some kind of synthesis of these approaches.

(A) 7.10. Transhipment under the Triangle Inequality

We would like to obtain results for transhipment analogous to those for the transportation problem (such as the existence of optimal solutions and ~~the existence~~ of potentials associated with unsurpassed solutions~~, etc~~). These results, however, seem quite hard to come by without making special assumptions.

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In this section we ~~shall~~ assume that the unit cost function $\underline{f}$ obeys the <u>triangle inequality</u>:

$$f(a_1,a_2) + f(a_2,a_3) \geq f(a_1,a_3),$$

for all a_1, a_2, $a_3 \in A$. Intuitively this states that there is no advantage to indirect shipment. To move from a_1 to a_3, one does not gain by going through a_2, and the same is true for circuitous routes involving several intermediate points. Now, if one compares the transhipment and transportation formulations, one sees that transhipment differs essentially in that it allows such circuitous shipments, while the transportation formulation forbids them. Since, under the triangle inequality, this extra freedom seems to do no good, one would expect that optimal solutions to the transportation problem ~~would turn out~~ ~~to~~ to be optimal for transhipment as well.

This expectation turns out to be correct, at least under certain limited circumstances. The key to the following proofs is the consideration of the dual function, the potential. We don't know if there is a more direct way of proving them.

~~Let~~ We us start with a relatively simple case. Given a measure space (A,Σ,ν), ν bounded, and a point $a_o \in A$, consider the problem of distributing a mass $\nu(A)$ concentrated at the point a_o over space A according to distribution ν. As a transportation problem on the product space $A \times A$ the problem

is trivial; in fact, there is exactly one feasible solution, 23
given by (1) below, and it of course must be optimal. As a transhipment problem there are many feasible solutions, but, under the triangle inequality, one feels intuitively that (1) should still be optimal. And so it is:

Theorem: Given bounded measure space (A, Σ, ν), and point $a_o \in A$, let $f: A \times A \to$ reals be bounded, measurable, obey the triangle inequality, and let $f(a_o, a_o) = 0$. Let measure λ^o on $(A \times A, \Sigma \times \Sigma)$ be given by

$$\lambda^o(G) = \nu\{a \mid (a_o, a) \in G\}, \tag{1}$$

all $G \in \Sigma \times \Sigma$. Then λ^o is best for the transhipment problem of minimizing

$$\int_{A \times A} f \, d\lambda \tag{2}$$

over bounded measures λ satisfying

$$\lambda' - \lambda'' = \nu_{a_o} - \nu. \tag{3}$$

(Here ν_{a_o} is the measure of mass $\nu(A)$ simply-concentrated on the point a_o).

Proof: One easily verifies that λ^o is feasible for (3). Define the function $p: A \to$ reals by

$$p(a) = f(a_o, a).$$

We will show p is bounded and a measure potential for λ^o. From the preceding duality theory this implies that λ° is best.

Bounded measurability of p follows from the corresponding properties of f. The triangle inequality implies that

$$p(a_2) - p(a_1) \le f(a_1, a_2) \tag{7.10.4}$$

~~(4)~~

for any a_1, $a_2 \in A$. Finally,

$$\lambda^\circ\left\{(a_1,a_2)\,\middle|\,p(a_2) - p(a_1) < f(a_1,a_2)\right\}$$

$$= \nu\left\{a\,\middle|\,p(a) - p(a_o) < f(a_o,a)\right\}, \tag{7.10.5}$$

~~(5)~~

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by (1). But $p(a_o) = 0$, and it is then clear that the set on the right side of (5) is empty: The common value in (5) is zero. This proves that p is indeed a bounded measure potential for λ°. ~~///~~

To make further progress we must introduce topology.

Definition: Let measure λ on $(A \times A, \Sigma \times \Sigma)$, and the function $p: A \to$ reals, be feasible for the (equality-constrained) transhipment problem and its dual, respectively. Let T be a topology on A.

Function p is a (transhipment) topological potential for λ iff the following condition is satisfied: If (a_1, a_2) is a point of support for λ, then

$$p(a_2) - p(a_1) = f(a_1, a_2).$$

("Point of support" refers to the topology $\mathcal{T} \times \mathcal{T}$ and ~~sigma~~ σ-field $\Sigma \times \Sigma$ on $\underline{A} \times \underline{A}$). That is, (4) is satisfied for all $(\underline{a}_1, \underline{a}_2)$, and is satisfied with equality for points of support.

This definition is appropriate for the equality-constrained variant of the transhipment problem. (For the inequality-constrained variant, an extra condition is needed, ~~namely~~ viz.: If $\underline{a}_o$ supports $\psi - (\lambda', \lambda'')$, then $\underline{p}(\underline{a}_o) = 0$; we ~~shall~~ do not discuss this, since it is not needed in what follows.)

Theorem: Let $\underline{p}$ be a (transhipment) topological potential for λ, for the equality-constrained transhipment problem, and let $\mathcal{T} \times \mathcal{T}$ have the strong Lindelöf property. Then $\underline{p}$ is a (transhipment) measure potential for λ.

The proof of this theorem is virtually identical with that of the corresponding theorem in the transportation problem (page ○○○), and will not be repeated here.

We are now ready for the next result, which generalizes ~~the~~ a preceding theorem at the cost of attaching some topological strings.

Theorem: Let μ, ν be bounded measures on $(\underline{A}, \Sigma)$. Let $\mathcal{T}$ be a topology on $\underline{A}$, such that $\mathcal{T} \subseteq \Sigma$, and $\mathcal{T} \times \mathcal{T}$ has the strong Lindelöf property. Let $\underline{f}: \underline{A} \times \underline{A} \rightarrow$ reals be bounded, continuous, measurable, ~~24~~ 28 and obey the triangle inequality, and let $\underline{f}(\underline{a}, \underline{a}) = 0$, all $\underline{a} \in \underline{A}$. Let measure λ° on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$ be best for the

transportation problem of minimizing (2) subject ~~ø~~to $\lambda' = \mu$, $\lambda'' = \nu$.

Then λ^o is best for the **transhipment** problem of minimizing (2) over bounded measures λ satisfying

$$\lambda' - \lambda'' = \mu - \nu.$$

Proof: This is trivial for $A = \emptyset$, so we may assume A is not empty. First ~~of all~~, it is clear that λ^o is feasible for the transhipment problem.

Since λ^o is transport optimal, $T \subseteq \Sigma$, and f is bounded continuous, it follows that λ^o has a bounded (transportation) topological potential: a pair of functions $p^o, q^o : A \to$ reals ~~which~~ that are bounded measurable, and for which

$$q^o(b) - p^o(a) \leq f(a,b), \qquad (7.10.6)$$

all $a, b \in A$, with equality in (6) if (a,b) supports λ^o. Now define the function $p : A \to$ reals by

$$p(a) = \inf\{p^o(x) + f(x,a)\}, \qquad (7.10.7)$$

the infimum being taken over all $x \in A$. (Note the distinction between p and p^o). We will show that p is a bounded **transhipment** topological potential for λ^o.

Boundedness of p follows from boundedness of p^o and f (remember that $A \neq \emptyset$).

For fixed x, the right side of (7) is a continuous function of $a \in A$, since f is continuous. Then p, as the infimum of a collection of continuous functions, is upper semi-continuous. Since $T \subseteq \Sigma$, it follows that p is measurable.

Next, we ~~will~~ prove that p is dual feasible, i.e. ~~that is~~,

$$p(a_2) - p(a_1) \leq f(a_1, a_2) \qquad (7.10.8) \quad \cancel{(8)}$$

~~for~~ all a_1, $a_2 \in A$. For any $x \in A$, we have

$$p(a_2) \leq p^\circ(x) + f(x, a_2), \qquad (7.10.9) \quad \cancel{(9)}$$

by (7). Also, by the triangle inequality,

$$f(x, a_2) \leq f(x, a_1) + f(a_1, a_2). \qquad (7.10.10) \quad \cancel{(10)}$$

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Adding (9) and (10), and simplifying, we obtain

$$p(a_2) - f(a_1, a_2) \leq p^\circ(x) + f(x, a_1)$$

Taking the infimum over $x \in A$ on the right-~~hand~~ side, we obtain (8).

Finally, let (a_1, a_2) be a point of support for λ°; we ~~will~~ show that (8) is satisfied with equality. First, we show that, for any $a \in A$, we have

$$p^\circ(a) \geq p(a) \geq q^\circ(a). \qquad (7.10.11) \quad \cancel{(11)}$$

H18

The left inequality in (11) follows from

$$p^\circ(a) = p^\circ(a) + f(a, a) \geq p(a).$$

The right inequality in (11) follows from taking the infimum over $x \in A$ in

$$p^\circ(x) + f(x,a) \geq q^\circ(a),$$

which in turn derives from (6).

From (11) we obtain

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$$p(a_2) - p(a_1) \geq q^\circ(a_2) - p^\circ(a_1) = f(a_1, a_2). \qquad \text{(7.10.12)} \; \cancel{(12)}$$

The equality in (12) arises from the fact that (a_1, a_2) supports λ°. Equation (12) shows that (8) must be satisfied with equality.

This completes the proof that p is a transhipment topological potential for λ°. Since $T \times T$ has the strong Lindelöf property, p is also a transhipment measure potential for λ°. Since p is also bounded, λ° is best for the transhipment problem. |||

We conclude by using this result to establish a theorem on the existence of optimal solutions to the transhipment problem.

Theorem: Let μ be a signed measure on (A, Σ), with $\mu(A) = 0$. Let T be a topology on A such that T is separable and topologically complete and Σ is the Borel field of T. Let $f: A \times A \to$ reals be bounded, continuous and obey the triangle inequality, and let $f(a,a) = 0$, all $a \in A$.

Then ~~there exists~~ a best solution λ° exists to the transhipment problem of minimizing (2) over bounded measures λ satisfying

$$\lambda' - \lambda'' = \mu.$$

Proof: Consider the transportation problem with origin and destination spaces $(\underline{A},\Sigma,\mu^+)$ and $(\underline{A},\Sigma,\mu^-)$, respectively. By the results of section 7.4 ~~there exists~~ a best solution, $\lambda°$, to this problem. (Note that $\mu^+(\underline{A}) = \mu^-(\underline{A}) < \infty$).

The premises of the preceding theorem are also fulfilled. (Since T is being separable metrizable, it has a countable base, hence so does $T \times T$, hence $T \times T$ has the strong Lindelöf property.) Hence this $\lambda°$ is also best for the transhipment problem of minimizing (2) over bounded measures λ satisfying

$$\lambda' - \lambda'' = \mu^+ - \mu^- = \mu.$$

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~~This completes the proof.~~ $\square$

It would be interesting to know whether the triangle inequality premise may be dropped from this theorem.

7.11. The Skew Transhipment Problem

We ~~Let us~~ return for a moment to the ordinary transhipment problem, (6.1)–(6.2) ~~(1)–(2) of section 6~~, with a finite number of locations. x_{ij} and x_{ji} are the flows from location i to location j, and vice versa, respectively. Define y_{ij}, the net flow from i to j, by

$$y_{ij} = x_{ij} - x_{ji}, \quad (i,\ j = 1,\ldots,n). \qquad (7.11.1)$$

We note at once that y_{ij} need not be non-negative. In fact $y_{ij} = -y_{ji}$ for all $i, j = 1,\dots,n$. That is, if the numbers y_{ij} were arrayed in matrix form, they would form a skew-symmetric matrix.

(The term "net flow" has been used above in an entirely different sense in Section 7.7, viz. as the net amount entering or leaving a given set of locations. It was represented ~~above~~ by a measure, λ_1 or λ_2, on $(\underline{A},\Sigma)$. Here it refers to a net movement on a given set of pairs of locations. It will be represented below by a signed measure (or, more generally, a pseudomeasure) on the product space $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$. Hence no confusion should arise between these concepts.)

The basic transhipment constraint, (6.1) ~~(1) of section 6~~, takes on a simple form when written in terms of net flows, viz.: ~~namely,~~

$$y_{i1} + \dots + y_{in} \le \alpha_i \qquad (7.11.2)$$

$(i = 1,\dots,n)$. What about the objective function, (6.2) ~~(2) of section 6~~? Can it be written in terms of net flows? We distinguish two cases, depending on the nature of the "gross" flow pattern (x_{ij}). This pattern is said to have the no-cross-hauling property iff $\min(x_{ij}, x_{ji}) = 0$ for all pairs (i,j) $(i,j = 1,\dots,n)$. That is, there never occurs a positive flow in both directions between any pair of locations (in particular, $x_{ii} = 0$, all i). In the no-cross-hauling case, we can ~~it is easy to~~ see that (1) ~~above~~ can be solved for x, viz. ~~namely,~~ $x_{ij} = \max(y_{ij},0)$.

Hence, if we restrict ourselves to such flow patterns, the objective function, (6.2) ~~(2) of section 6,~~ can be written: Minimize the sum of

$$f_{ij} \max (y_{ij}, 0) \qquad (7.11.3)\ \cancel{(3)}$$

over all n^2 pairs (i,j), $(i,j = 1,\ldots,n)$.

If cross-hauling occurs, the objective function cannot be written in terms of (y_{ij}) alone. However ~~On the other hand~~, little is gained by allowing cross-hauling. For consider the following two possible situations:

1. ~~(i)~~ $f_{ij} + f_{ji} < 0$ for some pair (i,j). Then there is no optimal solution to the transhipment problem, because the cyclic positivity condition, (6.4) ~~(4) of section 6~~, is violated.

2. ~~(ii)~~ $f_{ij} + f_{ji} \geq 0$ for all pairs (i,j). Then there is no point to cross-hauling. If x_{ij} and x_{ji} are both positive for some pair (i,j), an equal reduction of both ~~of~~ these numbers by $\min (x_{ij}, x_{ji})$ preserves feasibility and reduces transport cost — or at worst leaves it unchanged.

The problem of finding a skew-symmetric flow pattern $(y_{ij} = -y_{ji})$ that satisfies (2) and minimizes (3) will be called the <u>skew formulation</u> of the (ordinary) transhipment problem. The intuitive advantage of this over the ordinary formulation is that it automatically focuses attention on the flows without cross-hauling, which are the only interesting ones. Also, it can be argued that the <u>net</u> flow (y_{ij}) is really what one is looking for in transhipment problems in any case.

Condition (2) gives the *inequality*-constrained variant; the *equality*-constrained variant is obtained~~, of course,~~ by substituting "=" for "$\leq$" in (2).

~~It turns out that~~ all the arguments above carry over very neatly to the *measure-theoretic* transhipment problem. The remainder of this section will be devoted to showing this in detail. First, we need a few new concepts.

Transposition on a product space $A \times A$ refers to the interchange of left and right. It will be denoted by ~~a star~~ "*". Thus, if G is a subset of $A \times A$, its *transpose* is the set

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$$G^* = \{(a_1,a_2) \mid (a_2,a_1) \in G\}.$$

(Just "reflect" G through the "diagonal".) Similarly, the *transpose* of a function $f:A \times A \to$ reals is given by:

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$$f^*(a_1,a_2) = f(a_2,a_1).$$

We now want to extend this concept to *set* functions. For this, the following simple result is needed.

Lemma: Let (A,Σ) be a measurable space. If $G \in \Sigma \times \Sigma$, then $G^* \in \Sigma \times \Sigma$.

Proof: Consider the class, $\mathcal{G}$, of all $G \in \Sigma \times \Sigma$ for which $G^* \in \Sigma \times \Sigma$. We show that $\mathcal{G}$ owns all measurable rectangles, and that it is closed under complementation and countable unions. This implies that $\mathcal{G} = \Sigma \times \Sigma$, and concludes the proof.

Let $E, F \in \Sigma$. $(E \times F)^* = F \times E \in \Sigma \times \Sigma$, hence $E \times F \in \mathcal{G}$.

Let $G \in \mathcal{G}$. $((A \times A)\backslash G)^* = (A \times A)\backslash G^*$; this last set belongs to $\Sigma \times \Sigma$, since G^* does, hence $(A \times A)\backslash G \in \mathcal{G}$.

Let $G_n \in \mathcal{G}$ for $n = 1, 2, \ldots$. $(G_1 \cup G_2 \cup \ldots)^* = G_1^* \cup G_2^* \cup \ldots$; this last set belongs to $\Sigma \times \Sigma$, since each G_n^* does; hence $G_1 \cup G_2 \cup \ldots \in \mathcal{G}$. ||| $\square$

D

Definition: Let $\sigma : \Sigma \times \Sigma \to$ extended reals be a set function whose domain is the product ~~sigma~~ σ-field $\Sigma \times \Sigma$. The transpose of σ is the set function σ^* given by

(7.1.1.4)

$$\sigma^*(G) = \sigma(G^*), \qquad \text{~~(4)~~}$$

for all $G \in \Sigma \times \Sigma$.

This is well-defined, by the lemma just proved. It is easily established that σ^* is a measure, or signed measure, iff σ is a measure, or signed measure, respectively. Also, ~~sigma~~ σ-finiteness of σ implies the same for σ^*.

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D

Definition: Let σ be a pseudomeasure on $(A \times A, \Sigma \times \Sigma)$. The transpose of σ is the pseudomeasure $\sigma^* = (\mu^*, \nu^*)$, where (μ, ν) is any representative of σ.

For this to be a sound definition, σ^* must not depend on the particular representative of σ ~~which~~ that is chosen. Let (μ_1, ν_1) be another representative, so that

$$\mu + \nu_1 = \nu + \mu_1$$

(equivalence theorem). This implies

$$\mu^* + \nu_1^* = \nu^* + \mu_1^*,$$

by (4), so that $(\mu^*,\nu^*) = (\mu_1^*,\nu_1^*)$, and the same σ^* results. Hence the definition is bona fide.

Note that λ^* and σ^* remain defined on the product space $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$, in contrast to λ', λ'', λ_1, λ_2 discussed previously, which are defined on $(\underline{A},\Sigma)$. Note also that double transposition restores the original: $(\underline{G}^*)^* = \underline{G}$, $(\sigma^*)^* = \sigma$, etc.

18 In terms of transposes we now define the "skew" concepts needed for the skew transhipment problem.

D

Definition: Let σ be a signed measure or pseudomeasure on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$. Then σ is skew iff

$$\sigma^* = -\sigma. \qquad (7.11.5) \quad \cancel{(5)}$$

For signed measure σ, (5) states that σ takes on values of opposite sign on sets ~~which~~ that are transposes of each other. It follows that, if $\underline{G} \in \Sigma \times \Sigma$ is a symmetric set (i.e., ~~that is,~~ $\underline{G} = \underline{G}^*$), then $\sigma(\underline{G}) = 0$. In particular, the universe set $\underline{A} \times \underline{A}$ is symmetric, so that $\sigma(\underline{A} \times \underline{A}) = 0$. Thus a skew signed measure must be bounded.

For skew pseudomeasures we have the following result.

Theorem: Let σ be a pseudomeasure on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$. Each of the following conditions implies the other four:

(i) σ is skew;

(ii) σ^+ and σ^- are transposes of each other;

(iii) σ has a representative (μ,ν) for which $\mu = \nu^*$;

(iv) σ has a representative (μ,ν) for which $\mu + \mu^* = \nu + \nu^*$;

(v) $\mu + \mu^* = \nu + \nu^*$ for every representative (μ,ν) of σ.

Proof: Obviously, (ii) implies (iii) and (v) implies (iv).

(iii) implies (i). Letting (μ,ν) be the representative of σ with property (iii), we obtain $\sigma^* = (\mu^*,\nu^*) = (\nu,\mu) = -\sigma$, which shows that σ is skew.

(i) implies (v). Letting $\sigma = (\mu,\nu)$, we obtain $(\mu^*,\nu^*) = \sigma^* = -\sigma = (\nu,\mu)$, and (v) follows from the equivalence theorem for pseudomeasures.

(iv) implies (ii). Letting (μ,ν) be the representative of σ with property (iv), we obtain

$$((\sigma^+)^*, (\sigma^-)^*) = (\mu^*,\nu^*) = (\nu,\mu) = (\sigma^-,\sigma^+). \qquad (7.11.6)$$

The middle equality of (6) arises from (iv) via the equivalence theorem; the left and right equalities arise from two different ways of writing σ^* and $-\sigma$ respectively.

The left and right pairs in (6) are both mutually singular; for if $(\underline{P},\underline{N})$ is a Hahn decomposition for σ, so that

$\sigma^+(N) = \sigma^-(P) = 0$, then $(\sigma^+)^*(N^*) = (\sigma^-)^*(P^*) = 0$; but $\{P^*, N^*\}$ is a measurable partition of $A \times A$, so the left pair is mutually singular. By the uniqueness of the Jordan form, it follows that $(\sigma^+)^* = \sigma^-$ and $(\sigma^-)^* = \sigma^+$, which is condition (ii).

We now have a closed circle of implications. |||

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In connection with condition (iii) of this theorem, ~~it should be~~ noted that (if Σ is non-trivial) not all representatives of a skew pseudomeasure satisfy $\mu = \nu^*$. For example, the zero pseudomeasure is skew, and its representatives are the pairs (μ,μ) for all ~~sigma~~ σ-finite measures μ. But $\mu = \mu^*$ is not true for all such measures.

We are now ready for the skew formulation of the transhipment problem. The latter comes in bounded and unbounded formulations, and each of these can be "skewed". The bounded problem becomes one of finding the best of a feasible set of skew signed measures (these must be bounded, as noted above); the unbounded problem becomes one of finding the best of a feasible set of skew pseudomeasures.

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We ~~shall~~ formulate the skew bounded problem first. Measurable space (A,Σ) is given, together with a ~~sigma~~ σ-finite signed measure μ on it (net capacity), and a measurable function $f: A \times A \rightarrow$ reals (unit transport cost). The problem is: to

Find a skew signed measure σ on $(A \times A, \Sigma \times \Sigma)$ satisfying

$$\sigma' \leq \mu, \qquad (7.11.7) \quad ~~(7)~~$$

and minimizing

$$\int \underline{f} \, d\sigma^{+}. \qquad (7.11.8) \; \cancel{(8)}$$

Here σ' is, as usual, the left marginal of σ. Integral (8) is an indefinite integral over $\underline{A} \times \underline{A}$, and "minimization" is taken in its usual meaning of (reverse) standard ordering of pseudo-measures. Note that the upper variation, σ^{+}, occurs in the objective function, rather than σ itself.

Problem (7) ~~and~~ – (8) may be compared with the skew formulation of the ordinary transhipment problem, (2) ~~and~~ – (3). ~~Indeed,~~ it is not difficult to show that (7) ~~and~~ – (8) reduce to (2) ~~and~~ – (3), respectively, in the special case when Σ is a finite ~~sigma~~ σ-field. The discussion there also provides a rationale for the particular form that (7) ~~and~~ – (8) take.

We ~~Now for~~ take up the skew unbounded problem. For this we need a new concept, that of the marginal of a pseudomeasure. This in turn is a special case of the following.

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Definition: Let $(\underline{B},\Sigma',\sigma)$ be a pseudomeasure space, $(\underline{C},\Sigma'')$ another measurable space, and $g:\underline{B} \to \underline{C}$ a measurable function. The pseudomeasure induced on $(\underline{C},\Sigma'')$ by $\underline{g}$ from σ is defined iff the measures, μ and ν, induced from σ^{+} and σ^{-}, respectively, are both ~~sigma~~ σ-finite. In this case (μ,ν) is the induced pseudomeasure.

Starting with the pseudomeasure space $(\underline{A} \times \underline{A},\ \Sigma \times \Sigma,\ \sigma)$, the left marginal – if it exists – is the pseudomeasure induced on the space $(\underline{A},\Sigma)$ by the projection $\underline{g}'(\underline{a}',\underline{a}'') = \underline{a}'$, according to this definition. Similarly, the right marginal is that induced by the projection $\underline{g}''(\underline{a}',\underline{a}'') = \underline{a}''$. We shall use the notation σ', σ'' for these respective marginals, so that

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$$\sigma' = \left[(\sigma^{+})', (\sigma^{-})'\right], \quad \sigma'' = \left[(\sigma^{+})'', (\sigma^{-})''\right]. \qquad \text{(7.11.9)} \; \text{~~(9)~~}$$

Again, σ' is defined iff $(\sigma^{+})'$ and $(\sigma^{-})'$ are both sigma-finite, and similarly for σ''. In the case where σ is a bounded signed measure, these marginals are all bounded; hence σ' and σ'' are always well-defined. In fact, one easily verifies that, in this case, σ' and σ'' are bounded signed measures coinciding with the usual marginal concepts:

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$$\sigma'(\underline{E}) = \sigma(\underline{E} \times \underline{A}), \quad \sigma''(\underline{E}) = \sigma(\underline{A} \times \underline{E}),$$

if displayed insert ¶ after middle comma

all $\underline{E} \in \Sigma$. Hence the σ' in (7) may be looked upon as a special case of the definition just given.

The skew unbounded problem may now be stated. It is precisely the same as the skew-bounded problem, (7)–(8), except that σ ranges over the set of skew pseudomeasures for which σ' exists and satisfies (7). (Also, the given "net capacity" signed measure μ in (7) is replaced by the given pseudomeasure ψ).

Let us contemplate these skew formulations. One possibly disquieting feature of ~~them~~ is that left and right appear to be treated asymmetrically: σ' must exist and satisfy a certain condition, but not σ''. But this is an illusion, as the following result indicates.

Theorem: Let σ be a skew pseudomeasure on the product space $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$. Then σ' exists iff σ'' exists, and, in this case,

$$\sigma' = - \sigma''. \qquad (7.11.10) \qquad (10)$$

Proof: Let $\underline{E} \in \Sigma$. Then

$$(\sigma^{+})'(\underline{E}) = \sigma^{+}(\underline{E} \times \underline{A}) = \sigma^{-}(\underline{A} \times \underline{E}) = (\sigma^{-})''(\underline{E}). \qquad (7.11.11) \qquad (11)$$

The middle equality arises from σ^{+} and σ^{-} being transposes. Thus we obtain $(\sigma^{+})' = (\sigma^{-})''$. Similarly, $(\sigma^{-})' = (\sigma^{+})''$. Hence the two pairs in (9) are interchanges of each other, and σ', σ'' — if they exist — are negatives of each other. Also σ' exists iff σ'' exists. $\square$

Equality (10) also holds in the special case where σ is a skew signed measure. Thus in both bounded and unbounded skew formulations one could just as easily have written things in terms of σ'' as of σ'.

We now want to relate the skew to the non-skew formulations. In discussing the ordinary transhipment problem, we noted the connection between skew flows and ordinary flows having the "no-cross-hauling" property. To carry this connection over to the measure-theoretic problem, we need a generalization of this property.

Definition: Measure λ on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$ has the **no-cross-hauling property** iff λ and its transpose λ^* are mutually singular.

One easily verifies the following: If Σ is a finite sigma-field, this property in effect reduces to the one mentioned above for ordinary transhipment: $\min(x_{ij}, x_{ji}) = 0$. Any translocation has the no-cross-hauling property (since λ has all its mass on a set $\underline{P} \times \underline{N}$, and λ^* on the disjoint set $\underline{N} \times \underline{P}$, $\{\underline{P}, \underline{N}\}$ being a partition of $\underline{A}$). The converse of this is false, and even in a three-point space one can find a non-translocation with this property (exercise).

Theorem: Given product measurable space $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$, let $\mathbb{L}$ be the set of all sigma-finite measures λ on it with the no-cross-hauling property; let $\mathbb{L}_1$ be the set of those $\mathbb{L}$-measures whose marginals λ', λ'' are sigma-finite; let $\mathbb{L}_2$ be the set of those $\mathbb{L}_1$-measures ~~which~~ that are bounded. Also, let Ψ be the set of skew pseudomeasures σ on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$; let Ψ_1 be the set of those Ψ-pseudomeasures for which the left marginal σ' exists; let Ψ_2 be the set of skew signed measures on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$.

Let $\underline{g}$ assign to each $\sigma \in \Psi$ its upper variation:

$$\underline{g}(\sigma) = \sigma^+. \qquad (7.11.12)\ \cancel{(12)}$$

Let $\underline{h}$ assign to each $\lambda \in \mathbb{L}$ the pseudomeasure (λ,λ^*):

$$\underline{h}(\lambda) = (\lambda,\lambda^*) \qquad \text{(7.11.13)} \; \cancel{(13)}$$

Then $\underline{g}$ and $\underline{h}$ both establish 1-1 correspondences between the three pairs — $\mathbb{L}$ and Ψ, $\mathbb{L}_1$ and Ψ_1, and $\mathbb{L}_2$ and Ψ_2 — and are inverses of each other:

$$\underline{g}(\underline{h}(\lambda)) = \lambda, \; \underline{h}(\underline{g}(\sigma)) = \sigma. \qquad \text{(7.11.14)} \; \cancel{(14)}$$

Finally, if σ and λ are corresponding members of Ψ_1 and $\mathbb{L}_1$, then

$$\sigma' = (\lambda',\lambda''). \qquad \text{(7.11.15)} \; \cancel{(15)}$$

Proof: First, we show that the ranges of $\underline{g}$ and $\underline{h}$ are contained in the proper sets. If $\sigma \in \Psi$, the transpose of σ^+ is σ^-, and of course σ^+, σ^- are mutually singular; hence σ^+ has the no-cross-hauling property: $\underline{g}(\sigma) \in \mathbb{L}$. If, in addition, $\sigma \in \Psi_1$, then $(\sigma^+)'$ and $(\sigma^-)'$ are sigma-finite. The latter equals $(\sigma^+)''$ (cf. (11)); hence σ^+ has sigma-finite marginals: $\underline{g}(\sigma) \in \mathbb{L}_1$. If, in addition, $\sigma \in \Psi_2$, then it is bounded, so σ^+ is bounded: $\underline{g}(\sigma) \in \mathbb{L}_2$. This proves that $\underline{g}$ maps things into the right sets.

If $\lambda \in \mathbb{L}$, then (λ,λ^*) is a skew pseudomeasure: $\underline{h}(\lambda) \in \Psi$. Suppose, in addition, that $\lambda \in \mathbb{L}_1$, so that λ', λ'' are sigma-finite. For any $\underline{E} \in \Sigma$, we have

$$\lambda''(\underline{E}) = \lambda(\underline{A} \times \underline{E}) = \lambda^*(\underline{E} \times \underline{A}) = (\lambda^*)'(\underline{E}).$$

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Hence $\lambda'' = (\lambda^*)'$, and the latter is ~~sigma-finite~~ σ-finite. (λ,λ^*) is the Jordan form of $\underline{h}(\lambda)$, hence $(\underline{h}(\lambda))'$ exists: $\underline{h}(\lambda) \in \Psi_1$. If, in addition, $\lambda \in L_2$, λ is bounded, so (λ,λ^*) is a signed measure: $\underline{h}(\lambda) \in \Psi_2$. This proves that $\underline{h}$ maps things into the right sets.

It remains only to establish (14) and (15). For any $\lambda \in L$,

$$\underline{g}(\underline{h}(\lambda)) = (\lambda,\lambda^*)^+ = \lambda,$$

since (λ,λ^*) is the Jordan form of $\underline{h}(\lambda)$. For any $\sigma \in \Psi$,

$$\underline{h}(\underline{g}(\sigma)) = (\sigma^+, (\sigma^+)^*) = (\sigma^+, \sigma^-) = \sigma,$$

since σ^- is the transpose of σ^+. Finally,

$$\sigma' = ((\sigma^+)', (\sigma^-)') = ((\sigma^+)', (\sigma^+)'') = (\lambda',\lambda''),$$

if σ, λ are corresponding members of Ψ_1, L_1. This yields (15).

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This long theorem has a very simple interpretation. Compare the skew bounded transhipment problem, (7)–(8), for instance, with the ~~non-skew bounded~~ problem that follows:

Find a bounded measure λ satisfying

$$(\lambda',\lambda'') \le \mu$$

and minimizing

$$\int f \, d\lambda.$$

If we add the additional constraint that λ has the no-cross-hauling property, we find that the mappings g or h, (12) or (13), establish a 1-1 correspondence between the set of measures λ feasible for this problem and the set of signed measures σ feasible for the preceding problem. Furthermore, the objective functions assign the same utility to corresponding λ and σ, since $\lambda = \sigma^+$. Thus these problems are equivalent to each other in a rather strong sense.

Similarly, the unbounded skew and non-skew problems are equivalent to each other in this sense, if we add the "no-cross-hauling" constraint to the non-skew problem. The feasible sets in the unbounded problems are subsets of Ψ_1 and L_1, respectively, just as they are subsets of Ψ_2 and L_2, respectively, in the bounded problems.

~~Note that~~ for the ordinary transhipment problem the mappings g and h take forms we have already encountered: g becomes $x_{ij} = \max(y_{ij}, 0)$, and h becomes $y_{ij} = x_{ij} - x_{ji}$.

Finally, we want to investigate the effects of restricting attention to flows with the "no-cross-hauling" property. For ordinary transhipment we pointed out that, if an optimal flow exists at all, ~~then~~ some flow without cross-hauling is optimal. This property carries over to measure-theoretic transhipment.

First consider the process of "reducing" the flow pattern (x_{ij}) by subtracting $\min(x_{ij}, x_{ji})$ from x_{ij} and x_{ji} if these are both positive. This leads to the "no-cross-hauling" flow whose value at (i,j) is $\max(x_{ij} - x_{ji}, 0)$. The following concept generalizes this operation.

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Definition: Let λ be a ~~sigma~~ σ-finite measure on $(A \times A, \Sigma \times \Sigma)$. The no-cross-hauling reduction of λ is the measure $(\lambda,\lambda^*)^+$. (That is, form the pseudomeasure (λ,λ^*), ~~and~~ then take its upper variation.)

#

To show that $(\lambda,\lambda^*)^+$ does, indeed, have no cross-hauling, note first that (λ,λ^*) is skew. The upper variation of this is obtained by applying the mapping g, (12), whose range was proved to lie in the set of "no-cross-hauling" measures. We also have $(\lambda,\lambda^*)^+ \leq \lambda$, by the minimizing property of the Jordan form.

We can, ~~in fact,~~ obtain an exact expression for the size of this reduction, which may be called the "cross-hauling".

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Definition: Let λ be a ~~sigma~~ σ-finite measure on $(A \times A, \Sigma \times \Sigma)$. The cross-hauling associated with λ is the measure κ given by

$$\kappa = \inf(\lambda,\lambda^*).$$

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The following results show that these definitions capture quite well the intuitive meaning of these concepts. (Subtraction of measures is defined in ~~chapter 3,~~ section 3.1; if κ is finite this reduces to ordinary ~~element-wise~~ subtraction of elements).

be a measure

D **Theorem:** Let λ on $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$ ~~be sigma-finite~~. The no-cross-hauling reduction of λ equals $\lambda - \kappa$. Also, λ has the "no-cross-hauling" property iff $\kappa = 0$.

D **Proof:** The no-cross-hauling reduction of λ is the upper variation of the Jordan decomposition of the pair (λ,λ^*), and this is known to equal $\lambda - \underline{\inf}(\lambda,\lambda^*)$ (~~chapter 3,~~ section 3 1). The second statement is simply a special case of the theorem that a pair (μ,ν) is mutually singular iff $\underline{\inf}(\mu,\nu) = 0$, which was proved ~~above~~ in section 7 7. Here $\mu = \lambda$, $\nu = \lambda^*$.

~~Note, by the way, that~~ this theorem holds for **any** measure λ, not merely for ~~sigma~~-finite measures. ~~In the~~ general ~~case~~, $(\lambda,\lambda^*)^+$ refers to the upper variation of the Jordan decomposition of (λ,λ^*), which is well-defined for any λ on $\underline{A} \times \underline{A}$.

The cross-hauling measure κ also has the property of being symmetric ~~— that is~~ i.e., $\kappa(\underline{G}) = \kappa(\underline{G}^*)$ for any $\underline{G} \in \Sigma \times \Sigma$. (This is easily established from the fact that λ and λ^* enter symmetrically into its definition.) This implies that the left and right marginals of κ are equal: $\kappa' = \kappa''$.

50 13 With these preliminaries established, we are ready for our final result. This generalizes the argument given for ordinary transhipment and says, in effect: In looking for an optimal flow, ~~one~~ we might as well confine attention to flows without cross-hauling. The theorem applies to both bounded and unbounded formulations, and to both equality- and inequality-constrained variants.[29] [25]

29

Theorem: If measure λ on space $(\underline{A} \times \underline{A}, \Sigma \times \Sigma)$ is best (or unsurpassed) for the transhipment problem, then its "no-cross-hauling" reduction is also best (or unsurpassed), respectively.

Proof: For convenience we use $-\underline{f}$ in place of $\underline{f}$ in the objective function, and treat the problem as one of maximization. We ~~shall~~ also find it convenient to treat the measures ~~which are~~ discussed (~~and~~ which are all σ-finite) as pseudomeasures, so that they may be subtracted freely even though they may be infinite.

Let λ be best for the transhipment problem. First ~~of all~~,

$$\left((\lambda,\lambda^*)^{+\prime}, (\lambda,\lambda^*)^{+\prime\prime}\right) = (\lambda',\lambda'') = ((\lambda + \kappa)', (\lambda + \kappa)''), \qquad (7.11.16)$$

the equalities being understood in the pseudomeasure sense. To prove (16), we first verify that all six measures appearing there are σ-finite. λ' and λ'' are σ-finite since λ is feasible; κ and $(\lambda,\lambda^*)^+$ are both $\leq \lambda$, hence their marginals are σ-finite, too. The right~~-hand~~ equality arises from $\kappa' = \kappa''$. Similarly, $(\lambda,\lambda^*)^+$ and λ differ by κ (by the preceding theorem), and the same argument establishes the left ~~hand~~ equality in (16). Since feasibility depends only on the value of the pseudomeasure formed from the marginals in this way, it follows that $\lambda + \kappa$ and $(\lambda,\lambda^*)^+$ are also feasible flows.

Since λ is best, we must have

$$\int(-\underline{f})\,d\lambda \geq \int(-\underline{f})\,d(\lambda + \kappa). \qquad (7.11.17)$$

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(Here "$\succcurlyeq$" is the preferred-or-indifferent relation for standard order). The same pseudomeasure may be added to both sides of (~~16~~ 17) without disturbing the order relation. Let us add $\int f\, d\kappa$ to obtain

$$\int (-f)\, d(\lambda-\kappa) \succcurlyeq \int (-f)\, d\lambda \qquad (7.11.18)\ (\cancel{17}\ 18)$$

But $\lambda-\kappa = (\lambda,\lambda^*)^+$, by the preceding theorem, and (~~17~~ 18) states that this measure is at least as preferred as λ. Since λ is best, so is $(\lambda,\lambda^*)^+$.

Next, let λ be unsurpassed for the transhipment problem. Since λ is feasible, ~~hence~~ so is $(\lambda,\lambda^*)^+$, by (~~15~~ 16). Suppose that $(\lambda,\lambda^*)^+$ is surpassed by some feasible measure ν:

$$\int (-f)\, d\nu \succ \int (-f)\, d(\lambda,\lambda^*)^+ \qquad (7.11.19)\ (\cancel{18}\ 19)$$

Adding $\int (-f)\, d\kappa$ to both sides of (~~18~~ 19), we obtain

$$\int (-f)\, d(\nu+\kappa) \succ \int (-f)\, d\lambda \qquad (7.11.20)\ (\cancel{19}\ 20)$$

Since ν is feasible, the same argument leading to (~~15~~ 16) establishes that $\nu + \kappa$ is feasible. (~~19~~ 20) then states that λ is surpassed by $\nu + \kappa$. This contradiction proves that $(\lambda,\lambda^*)^+$ is unsurpassed. ▫

FOOTNOTES - CHAPTER 7

629 [1]A. M. Faden, "The Abstract Transportation Problem," ~~pages~~ pp. 147-175 ~~of~~ in Papers in Quantitative Economics, vol. 2, A. M. Zarley, editor (University Press of Kansas, Lawrence, 1971), is a less advanced version of sections 7.1 ~~through~~ to 7.5.

630 [2]S. Vajda, Readings in Mathematical Programming, 2nd ed. (Wiley, New York, ~~2d ed,~~ 1962); G. B. Dantzig, Linear Programming and Extensions (Princeton University Press, Princeton, 1963).

634 [3]The measure induced by a ~~sigma~~ σ-finite measure is always abcont, even if not ~~sigma~~ σ-finite. ~~See page~~

636 [4]There are certain complications if mixtures of equality and inequality constraints appear within the capacity block, (1.1). ~~(1) of section 1.~~ We ~~shall~~ do not discuss these.

638 [5]First proposed by G. B. Dantzig, "Application of the Simplex Method to a Transportation Problem," ~~chapter~~ ch. 23 ~~of~~ in Activity Analysis of Production and Allocation, T. C. Koopmans, ed., ~~(ed),~~ (Wiley, New York, 1951), ~~pages~~ pp. 361-362.

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639 [6]For further information see, ~~for example~~ e.g., J. L. Kelley, General Topology (Van Nostrand, Princeton, 1955).

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660 [7]For topologies, the process of generation can be written in two steps, as just indicated. For ~~sigma~~ σ-fields it cannot be written in even a ~~finite~~ countable number of steps. In both cases, however, the basic concept is the same: the intersection of all topologies (respectively, ~~sigma~~ σ-fields) containing the given class $\mathcal{G}$.

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662 [8]One can show that this definition reduces to that of chapter 5 for the special case discussed there where ($\underline{A}$,T) is an interval of the real line with the usual topology.

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662 [9]If $\underline{A}$ is the real line with the usual topology, one can show that its Borel field as here defined coincides with "Borel field" as defined in chapter 2. The same is true for $\underline{n}$-space.

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669 [10]~~A number of~~ results from the theory of weak convergence of measures are used in this part of the proof. On this theory see P. Billingsley, Convergence of Probability Measures (Wiley, New York, 1968), Chap~~ter~~ ~~I~~ 1, and K. R. Parthasarathy, Probability Measures on Metric Spaces (Academic Press, New York, 1967).

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11. Ulam's theorem; see Billingsley, *Convergence*, pp. 5–6.

12. Billingsley, *Convergence*, p. 32.

13. Billingsley, *Convergence*, pp. 30–31.

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14. ~~11~~ Billingsley, *Convergence*, page 9, has a similar theorem, but with equalities in place of inequalities; a simple twist of his proof yields the statement just made.

15. Parthasarathy, *Probability Measures*, pp. 29–30.

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16. ~~12~~ For readers familiar with general topology the following remarks will serve to "place" this concept. The strong (or "hereditary") Lindelöf property is implied by the possession of a countable base, and in turn implies the (weak) Lindelöf property that every covering of the space by open sets contains a countable subcovering. One shows by counterexamples that neither of these implications can be reversed. But in a metrizable space these three properties are logically equivalent, and also equivalent to separability. See A. Wilansky, Topology for Analysis (Ginn, Waltham, Mass., 1970). This book ends with a remarkable table of implications among topological properties.

Note also that T', T'' strong Lindelöf does not guarantee that $T' \times T''$ is even weak Lindelöf (e.g. the RHO or Sorgenfrey topology on the real line).

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17. ~~13~~ $T' \subseteq \Sigma'$ and $T'' \subseteq \Sigma''$ does not guarantee that ~~do not necessarily imply~~ $(T' \times T'') \subseteq (\Sigma' \times \Sigma'')$, nor need we make this stronger assumption.

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18. ~~14~~ In fact, the single extra premise ~~that~~ $f > 0$ implies that $\lambda^{o''} = \mu''$, hence (see below) that a wide-sense topological potential exists. But the proof in this case is more complicated.

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705 ¶ 19. [15]L. Kantorovitch, "On the Translocation of Masses", Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS, vol. 37, No. 7-8, 199-201, 1942 (reprinted in Management Science, 5:1-4, October, 1958).

709 ¶ 20. [16]A. Orden, "The Transhipment Problem", Management Science, 2:276-285, April, 1956. We have adopted Orden's spelling of the word "transhipment."

710 ¶ 21. [17]We ~~shall~~ do not deal with nonlinear objective functions in this chapter, although some of our results do generalize to this case. Note that a nonlinear objective can still be pseudo-measure-valued, as in chapter 5 ~~above~~.

712 ¶ 22. [18]Cf. P. A. Samuelson, "Intertemporal Price Equilibrium: A Prologue to the Theory of Speculation," Weltwirtschaftsliches Archiv, 79:181-221, 1957; C. H. Kriebel, "Warehousing with Transhipment Under Seasonal Demand," Journal of Regional Science, 3:57-69, Summer, 1961.

714 ¶ 23. [19]Orden, "The Transhipment Problem", ~~pages~~ pp. 283-285. There are several algorithms for finding a shortest route, ~~for example~~ e.g., G. B. Dantzig, "On the Shortest Route Through a Network," Management Science, 6:187-190, January, 1960.

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[20] The most fruitful network problems have not been of the transhipment type (1)-(2), but of the following form: given a flow capacity on each link of a network, maximize the flow from a given source to a given sink. See L. R. Ford, Jr., and D. R. Fulkerson, Flows in Networks (Princeton University Press, Princeton, 1962). A measure-theoretic treatment of these problems can be given, but we ~~shall~~ not do so in this book.

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[21] The terminology (but not the meaning) is from Kantorovitch.

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[22] M. Beckmann, "A Continuous Model of Transportation," Econometrica, 20:643-660, October, 1952; "The Partial Equilibrium of a Continuous Space Market," Weltwirtschaftliches Archiv, 71:73-87, (1953).

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[23] Exercise: prove this uniqueness assertion, without making the assumption that $\{a_o\} \in \Sigma$.

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[24] ~~Actually~~, the properties of T imply that $(T \times T) \subset (\Sigma \times \Sigma)$; hence the measurability of f follows from its continuity. The same is true in the following theorem.

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[25] We have been stating constraints in the inequality form in this section. But if "$\leq$" is replaced by "=" in these formulas, the discussion is still valid word for word.
