~~CHAPTER~~ 6. MARKETS

This chapter will take up two major topics: the set of feasible trading possibilities open to an individual, and the equilibrium of markets — in particular, the real-estate market. The first topic continues the feasibility discussion of chapter 4, budget constraints being special kinds of institutional constraints. Budget constraints merit special treatment because of their peculiar form and great importance generally.

Our aim in this chapter is to bring these concepts within our measure-histories-activities framework. We ~~shall~~ touch on only a few high points and special cases. (A comprehensive treatment is out of the question, since most of economics could be encompassed in this chapter).

6.1. Budget Constraints

Consider the balance sheet of a given person, firm, or other agent, at time t. The major items may be classified into physical assets, financial assets, liabilities, and net worth. Physical assets include all things owned by the agent in question, such as land, buildings, equipment, inventories, household goods, ~~etc.~~ Financial assets include all monetary claims on other agents, such as cash (a claim on the banking system), accounts receivable, bonds, promissory notes, ~~etc.~~ (Corporate stock is a borderline asset. In closely held firms

~~one~~ we can think of the stock as representing the net assets of the corporation itself; in large publicly-owned companies it functions more like a debt with uncertain face value, and should perhaps be classified as a financial asset). **Liabilities** are the claims against the agent in question by other agents. Thus every liability on one balance sheet is matched by a financial asset on some other balance sheet, and vice versa. The totality of all financial assets on all balance~~d~~ sheets combined (including government bodies, churches, universities, etc., and all foreigners as well) should equal the totality of all liabilities, if no slipups in accounting have been made.

Finally, **net worth** equals the value of physical assets plus financial assets minus liabilities. It follows that the totality of all net worths on all balance sheets combined should equal the totality of the values of physical assets on all balance sheets, since total financial assets cancel against total liabilities.

~~A few~~ we comments briefly on this scheme. The neat dichotomization of assets becomes a little ragged upon examination. Actually, nearly all assets represent claims of one sort or another — in particular, claims against "trespass" in a generalized sense: To own a commodity means that no one else has the right to use it. It is a simplifying idealization to substitute the commodity itself for the bundle of rights and claims entailed by its ownership. ~~There are~~ a number of "intangible" assets which do not fit easily into either the physical or financial category ~~- for example,~~ (e.g. patents, franchises, easements) ~~-~~ but may be expressed in terms of claims and rights. "Goodwill" is not even a claim, but is a reflection of the habits of trading partners.

~~There are~~ also, a number of items not customarily included among assets ~~which~~ perhaps should be. These include government-owned resources ~~which are~~ placed at public disposal free or for a nominal fee (e.g., parks, roads, police and fire protection, the judicial system, ~~etc.~~ The person himself (and perhaps some dependents) might be included among his physical assets: he owns his own body.

Finally, there is the important category of <u>control</u> of assets, as opposed to ownership of assets. This includes <u>rentals</u> (of land, labor, etc.) and the holding of office, and will be discussed further below.

Having discussed these complications briefly, ~~let us~~ we go to the opposite extreme and simplify the balance sheet for purposes of analysis. We assume there is just one homogeneous kind of financial asset (call it "bonds") ~~- which~~ that accumulates interest at rate $k(t)$ at time t. (Interest is compounded continuously; $k(t)$ is assumed to be continuous and nonnegative).

Focusing on one economic agent, let $b(t)$ be his <u>net bondholding</u> at time t. This is defined as financial assets minus liabilities. If $b(t) > 0$, the agent in question is a net creditor at time t; if $b(t) < 0$, the agent is a net debtor. (The sum of $b(t)$

over all agents must be identically zero for any time t).

Three influences are assumed to change b over time: the accumulation of interest, the sale of physical assets (which raises b), and the purchase of physical assets (which lowers b). Let $f_1(t)$, $f_2(t)$ be the rate at which physical assets are being sold and purchased, respectively, at time t, in dollar terms. (These are assumed to be continuous, non-negative functions for the time being).

We then have the differential equation:

$$Db(t) = k(t)b(t) + f_1(t) - f_2(t). \qquad (1)$$

(6.1.1)

The only term that needs comment in (1) is the interest term $k(t)b(t)$. This has the sign of $b(t)$, indicating that interest payments are positive for creditors and negative for debtors, so that (1) is correct for both these cases. ~~Note that,~~ realistically, there should be several other terms on the right side: wages and rentals, taxes and transfers, etc. These are all being ignored for simplicity's sake.

~~First of all,~~ a simple transformation allows us to get rid of the interest term in (1). Define **discounted net bond-holding** to be the function $b' : T \to$ reals given by

$$b'(t) = b(t)e^{-\int_0^t k(t)dt}. \qquad (2)$$

(6.1.2) zero

The integral in (2) is the ordinary Riemann integral. If $t < 0$, the standard convention of elementary calculus, that zero $\int_\theta^t = -\int_t^\theta$, is followed. Similarly, ~~one~~ we define~~s~~ discounted sales and purchases, f_1' and f_2', by (2), with f_1, f_2 in place of b, respectively. Equations (1) and (2) then imply that

$$Db'(t) = f_1'(t) - f_2'(t), \qquad \text{(6.1.3)} ~~(3)~~$$

so that the interest term drops out. This simplification is useful, and we ~~shall~~ use discounted values wherever possible. (If there are multiple interest rates, or if the rate varies with b, this simplification is not available).

We have taken sales and purchases to be rates ~~which~~ that are continuous functions of Time. Let us now generalize this. Transactions occur not only (if at all) continuously, but also in lumps. To incorporate this possibility, we take sales and purchases to be (bounded) measures, λ_1 and λ_2, on universe set T. Thus $\lambda_1(G)$ = value of sales in period G, for any Borel set G on the real line. We can use either current dollars or discounted dollars as our measurement units; for simplicity we use discounted dollars.

The differential equation (3) — (or rather, its integral) — then generalizes as follows. For any two moments, t' and t", with t' < t", we have

$$b(t'') - b(t') = \lambda_1[t', t'') - \lambda_2[t', t''). \qquad \text{(6.1.4)} ~~(4)~~$$

Here [t', t") is the time-interval between t' and t", including the past endpoint t', but excluding the future endpoint t". b(t) is discounted net bondholding (the prime has been dropped).

~~There is~~ a conventional element is involved in (4). If a "lumpy" transaction of positive measure occurs at the point *t*, it is arbitrary whether *b*(*t*) is defined ~~so as~~ to *include* or *exclude* the transaction at *t* itself. According to (4) it *excludes* this transaction, and this makes *b* continuous from the past, but not necessarily from the future.[1]

~~Let us~~ We now resolve the sale and purchase measures ~~– which are~~ (given in dollar terms) – into *prices* and *quantities*. That is, total sales will be a composite of sales in different markets at different prices. But what is a market? At the least, to identify a market one ~~has to~~ must know *what* is being sold, *where* it is being sold, and *when* it is being sold. This gives a triple (r,s,t), and suggests that markets be identified with points of $R \times S \times T$. The set of all markets will then be a subset $E_o \subseteq (R \times S \times T)$. Point (r,s,t) will belong to E_o iff the resource *r* is being sold at location *s* at time *t*. We assume that $E_o \in (\Sigma_r \times \Sigma_s \times \Sigma_t)$.

We ~~shall~~ first make the competitive assumption that a single ruling price prevails in each market. That is, ~~there is~~ a function $p: E_o \to$ reals exists, where $p(r,s,t)$ ~~being~~ is the *price* at which resource *r* is sold at location *s* at time *t*. *p* is assumed to be measurable. In what follows we ~~shall~~ also assume it to be non-negative, although negative prices (for "illth", or noxious "resources") can easily be handled. Here, as above, we distinguish between *current prices*, p ~~(those at which sales~~ (those at which sales actually occur), and *discounted prices*, p':

break only at =

exp[—]

$$p'(r,s,t) = p(r,s,t)e^{\left[-\int_0^t k(t)\,dt\right]}$$

The sales (or exports) of ~~the~~ an agent over the set of markets will be given by a measure, μ_1, on universe set E_o. μ_1 is in terms of physical quantities, not dollar values, and $\mu_1[E_o \cap (F \times G \times H)]$ is the total mass of resources of types F sold in region G in period H. The value sold over various resource-types, regions, and periods may now be expressed as an indefinite integral over universe set E_o:

$$\int p\, d\mu_1. \qquad \text{(6.1.5)} \; \cancel{(5)}$$

The value measure (5) is in either current or discounted dollars, depending on whether p is current or discounted prices. For simplicity, ~~let us~~ take (5) to be in discounted terms. The relation between (5) and the value of sales measure λ_1 is then given by:

$$\lambda_1(G) = \int_{[E_o \cap (R \times S \times G)]} p\, d\mu_1 \qquad \text{(6.1.6)} \; \cancel{(6)}$$

subscript to $\int$

for any $G \in \Sigma_t$. Equation (6) simply states that the (discounted) value of total sales in time-period G is that over all Resources and all Space in that period. (If (5) is extended to universe set $R \times S \times T$ by defining it to be zero on $(R \times S \times T) \setminus E_o$, then λ_1

is ~~simply~~ the marginal on component space T of this extended measure).

Similarly, there will be a (physical quantity) purchase (or import) measure μ_2 over E_o, leading to a value of purchase measure (5) over E_o, whose relation to λ_2 is given by (6) (μ_1 being replaced by μ_2 in (5) and (6), ~~of course~~).

The analysis ~~up~~ to this point consists essentially in having set out a number of accounting identities, and no constraints on the actions of the agent have yet been mentioned. The problem may be expressed as follows. What combinations of sale and purchase measures, (μ_1, μ_2) are financially available to the agent? Or, in short, what are the exchange possibilities?

There are many possible answers to this question, depending on institutional arrangements – in particular, on the structure of the capital market.

One simple and popular – though not very realistic – approach is to assume perfect information, including a knowledge of the time, t^o, at which the agent in question will die. The constraint then takes the simple form:

$$b(t^o) \geq 0. \qquad (6.1.7) \; ~~(7)~~$$

That is, the agent must have repaid all debts by the time he expires; or, more exactly, the amount owed to him must be at least as large as the amount he owes to others at this time.

We ~~Let us~~ express (7) in terms of the sales and purchase measures, μ_1 and μ_2. Consider the options open to the agent at time $t_o < t^o$, when he starts with an initial net bond holding of $b(t_o)$, which may be negative (starting in debt), positive, or zero. Substituting from (6) and (4) into (7), we obtain

$$b(t_o) + \int_{[E_o \cap (R \times S \times [t_o, t^o))]} p \, d\mu_1 \geq \int_{[E_o \cap (R \times S \times [t_o, t^o))]} p \, d\mu_2. \qquad \text{(6.1.8)} \; \cancel{(8)}$$

which $\cancel{(8)}$ has a very simple interpretation. The right-hand term is the discounted value of all purchases made in the interval $t_o \leq t < t^o$; the middle term is the discounted value of all sales in that interval. (8) then states that the present value of purchases cannot exceed the present value of sales plus (the present value of) the initial credit balance. This is a direct generalization of the familiar linear consumer budget constraint:

$$y \geq p_1x_1 + \ldots + p_nx_n,$$

where the "income" term y may be interpreted as net credit, and x_i is net purchases of commodity i; (if $x_i < 0$ this indicates a sale rather than a purchase).

The form of (8) allows a lumpy transaction to occur at the initial point, t_o, but not at the point of expiration, t^o. This is an artifact of our definitions, and could be altered if desired by a minor modification of (8).

The constraint (7) or (8) is attractive because of its linearity, but rather tenuous from the viewpoint of realism. There is, first, ~~of all,~~ the problem of what to do with entities that ~~which~~ do not have a natural lifespan, such as corporations or government bodies. But even apart from this, this constraint allows arbitrarily high indebtedness at any time before t^e, which is clearly untrue of any existing credit system.

A better approximation to reality is obtained by introducing *collateral* requirements. These allow indebtedness (negative bondholding) up to a point determined by ~~one's~~ other assets. The other balance sheet categories must now be taken into account.

~~Let us~~ Suppose ~~that~~ the agent's physical assets have been appraised and that $v(t)$ is the *value of physical assets* at time t. Also let $w(t)$ be his *net worth* at that time. The basic balancesheet identity is

$$w(t) = v(t) + b(t) \qquad (6.1.9)\ \ (9)$$

This can be measured in either current or discounted dollars.

A simple form of collateral constraint is then

$$-b(t) \geq cw(t) \qquad (6.1.10)\ \ (10)$$

for all $t \in T$, where c is some positive real constant. That is, one may ~~is allowed to~~ go into debt up to some multiple of one's net worth. From (9), (10) can also be written in the equivalent form

$v(t)\,c/(1+c)$

$$-b(t) \geq \frac{c}{1+c} v(t), \qquad \cancel{(11)}\ (6.1.11)$$

all t, so that one can borrow a fraction of every dollar's worth of physical assets.

Still more realistic would be a condition ~~that~~ ~~takes~~ taking account of the fact that physical assets vary considerably in their ability to serve as collateral. Best of all is real estate, which is easily appraised, durable, and ~~which~~ cannot be absconded with. On the other hand, "human capital" — (the value of a person's own body as measured, say, by the discounted value of net future earnings) — is poor collateral, because it is hard to appraise, and ~~because~~ its mobility and long payback period make repayment difficult to enforce. For this reason students find it difficult to obtain unsecured long-term educational loans. (Under other institutional conditions human capital could function well as collateral; e.g., consider ~~think of~~ indentured servitude. ~~for example~~).

This realistic complication could be represented by replacing the right side of (11) with a weighted sum, each class of assets multiplied by the appropriate fraction corresponding to its collateral-serving ability.

Special kinds of economic agents have special kinds of budget conditions constraining them. Government bodies are limited by legislative appropriations, banks by reserve requirements. (To express the latter, one must distinguish the various categories of financial assets and liabilities; it will no longer do to lump them together as "bonds" as we have been doing.)

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6.2. Rentals

We now add the possibility of rental transactions to those of sale and purchase. Rentals are very important, much more so than ~~one would gather~~ might be inferred from their modest share of national income.

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Abstractly, a rental transaction occurs when one relinquishes control, but not ownership, ~~over~~ of an object. The most important type of rental by far is the employment relation, in which the worker places himself (within limits) at the disposal of the employer without relinquishing ownership ~~over~~ of his own body; ~~that is~~ i.e., without becoming the employer's slave. Then we have real-estate rentals, leading to the ordinary landlord-tenant relation. And there are a large number of miscellaneous rental markets, e.g., for cars, furniture, machinery, costumes, ~~etc.~~

A number of points need clarification. First, what is "ownership" and what is "control"? We are not concerned here with any strict legal definitions, but with the functional concepts as they relate to the set of feasible options open to an agent.

To control an object for a certain time-interval, as the term is used above, means to secure the acquiescence of other people not to interfere with one's use of the object, or ~~to~~ try to use it themselves. In the employment relation the "object" is another person, and the relation entails a willingness to obey orders within a certain "legitimate" range.

Ownership may now be defined in terms of control. To <u>own</u> an object means either to have permanent control of it, or — in case it is rented out — to re-acquire permanent control at some stipulated future date. In brief, the owner of an object is the agent to whom control ultimately reverts.

~~As usual,~~ realistic complications cloud these neat concepts. Control is a matter of degree. Much of the legal system consists of restrictions on the uses to which an agent can put the objects he owns or rents. Restrictions are especially important in the case of rentals, for the owner will rarely relinquish control without stipulating limits on the uses to which the rented object is to be put. If nothing else, the owner has an interest in the maintenance of the rented object, since it will eventually revert to his own use.

The essence of the rental relationship, then, is <u>serial control</u>. The owner contracts with someone to give up control of the object for a limited time in exchange for a rental payment. (In the employment relation, this of course is the wage). There may be a whole sequence of such renters with whom the owner contracts in turn, as when a worker moves from job to job, or a landlord rents to tenant after tenant. Another possibility — if this is permitted by the rental contract — is for the renter ~~is~~ in turn to relinquish control to a third party, giving a <u>subleasing</u> arrangement; he in turn could rent to a fourth party, etc.[2]

The same object may concurrently be changing ownership through outright sales. The times ~~at which~~ ownership changes hands need not coincide with the times ~~at which~~ control changes hands. Finally, ownership or control at any stage can be exercised jointly, with power centered in some committee of separate interests. The pattern of control can become rather tangled.

~~In one limiting case the ownership relation recedes until it essentially disappears for practical purposes. Suppose there is an <u>infinite</u> sequence of changes of control. In this case, there is no agent to whom control ultimately reverts, and hence, in effect, no owner, whatever the legal situation.[3] (Also see footnote 3 below).~~

~~Let us~~ We briefly consider the relation between rentals and <u>services</u>. According to our previous discussion ~~(pages)~~ (Section 2.7) a service activity is one in which the histories ~~which~~ that enter into the activity are owned by different agents. Now if <u>B</u> rents an object he owns to <u>A</u> (the "object" may be <u>B</u> himself as an employee), and <u>A</u> uses this object (together with others that <u>A</u> owns) to run an activity, we may speak of <u>B</u>'s providing a service for <u>A</u>. Thus, a worker provides labor services, a landlord provides housing services, etc. The "rental" and the "service" are just two aspects of the same relation, the former concentrating on the transaction ~~which~~ that brings the factors together, the latter on the activity in which the factors jointly participate.

~~On the other hand,~~ However, rental is not the only way ~~in which~~ diversely owned factors are brought together. The issue revolves around which agent is in control of the process, the recipient of service, A, or the provider, B. Consider the service of watch repairing. Here the owner, A, surrenders control of his watch to the repairman, B. But far from receiving a rental payment for this surrender of control, A actually pays B for the service rendered.

The relation in which A and B stand in this case is not that of employer and worker, or tenant and landlord, as in the case of rentals, but that of *bailor* and *bailee*, respectively. Without worrying about the legal niceties involved, ~~let us~~ we refer to the general relation ~~which~~ that obtains here as a *bailment* relation. In bailments, A relinquishes control of an object he owns to B, B performs a service ~~which~~ that benefits the object, B returns the object to its owner A, and A pays B for the service. In rentals, it is B who relinquishes control to A, who uses the object for his own benefit, and again pays B for this service.

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Bailment relations are very common, perhaps almost as common as rentals. Most repair services are bailments, including "repairs" to A himself, as by surgeons and barbers. Storage services provided by warehouses, and transportation services provided by postmen, or by common carriers for a person or his goods, are other examples.

The large publicly-owned corporation may be thought of as involving a bailment relation, in which the physical assets of the corporation are turned over to the control of management, by the stockholders collectively.

We have so far divided services into rental and bailment types, depending on who is in control. But, ~~as discussed above,~~ control is a matter of degree, and there are borderline cases where one type blends into the other. An unskilled worker and a surgeon both provide services; the first seems clearly involved in a rental relation, the second in a bailment. At skill levels intermediate between these two the control pattern will shift gradually from one of these forms to the other. We thus get situations of shared control.

Finally, consider social activities such as parties, picnics, and beach outings, ~~etc.~~, where the various participants provide each other with "companionship" services. These do not fit either of the categories above, and control itself — (in the sense of a single agent coordinating the factors entering the activity, without interference from any other agent) — may not exist.

Having examined some institutional features of rental and related markets, ~~let us~~ we turn to the problem of the determination of rental prices.

Assuming competitive markets, one's first impulse is to imitate the structure of the sales market and postulate a rental price function, π, whose domain is a subset of $\underline{R} \times \underline{S} \times \underline{T}$; $\pi(\underline{r},\underline{s},\underline{t})$ is the rental price for resource-type $\underline{r}$ at location $\underline{s}$ at time $\underline{t}$. Function π would have the dimension of money per unit mass <u>per unit time</u> (e.g., wage in dollars per man-hour, land rent in dollars per acre-year, ~~etc.~~).

This may be a fair approximation, and many rental markets appear to have a structure resembling this. But it has one basic shortcoming, viz. ~~namely,~~ that the rent does not depend on how the resource is going to be used by the renter. ~~Now,~~ since the owner will eventually regain control of his property, he will not be indifferent between uses that ~~which~~ leave his property dilapidated, ~~which~~ leave it unaffected, or ~~which~~ enhance its value. On the contrary, a premium would be required for him to rent to someone who will dilapidate his property, while he will be willing to accept a lower rent from someone who will return his property in improved condition. Indeed, if the improvement is big enough, he will be willing to accept a <u>negative</u> rent, — i.e. ~~that is~~, to pay to have his property used by the other person. (In this case the direction of service is reversed, and the rental relation has in fact become a <u>bailment</u> relation).

Thus, a worker would be willing to accept a lower wage on a job that ~~which~~ affords training opportunities than on one that ~~which~~ does not.[5] If the training opportunities were sufficiently rich, he might even accept a negative wage (we would probably call this "going to school" rather than "working"; the borderline between these cases is not sharp).

"Dilapidation" and "enhancement" refer to the position of the rented object in Resource-space, R, upon revision to the owner. But the same analysis applies to physical Space, S. A car-rental agency will demand a premium payment from someone who wants to return the car at some out-of-the-way place.

The potential renter, furthermore, may be concerned not only with the endstate of the rented object, but with the entire time-path over which it moves and the activity in which it participates. In the employment relation, the worker will be concerned with the pleasantness or unpleasantness of working conditions, and require a premium for working under poor conditions. The landlord may require that his premises not be used for certain disapproved activities, or at least that he be paid a premium if they are.

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<u>Discrimination</u> may be considered a special case of preferences concerning alternative activities into which one's rented property enters. It refers to preferences among alternative individuals or types of people ~~who are~~ participating in these activities. "Discrimination" <u>per se</u> refers to a preference for non-association with someone, while "nepotism" refers to a preference for association.[5] We speak of this as a special case of preferences over activities, because the definition of "activity" gives the distribution of mass over the entities participating in it, and so will specify the types of people involved. One may discriminate, in the first instance, with reference to ~~one's~~ a trading partner, and secondly, with

reference to that person's ~~the people he is~~ associates ~~with~~.

Discrimination exists also on the sales market, but it is probably more important in relation to rentals because ~~. The reason is that~~ the association between trading partners is closer and lasts longer when rentals are involved, and attraction or aversion is therefore likely to be more salient. For this reason we omitted any discussion of discrimination in connection with outright sales.

~~In summary,~~ it would appear that any model of the rental market that ~~which~~ postulates a rental function $\pi(r,s,t)$ is inadequate. (The cases where such a rental function does obtain seem to be those where the uses ~~to which~~ of the rented object ~~will be put~~ are so circumscribed, by custom or by explicit agreement, that one need not be concerned by their variation.)

What, then, does an adequate representation of the rental market look like? The following model incorporates some of the considerations discussed above. It makes rentals depend, not on points of $R \times S \times T$, but of $(R \times S \times T)^2$. For $t_1 < t_2$, $\pi(r_1,s_1,t_1,r_2,s_2,t_2)$ is the price to be paid for attaining control of a unit of resource r_1 at location s_1 at time t_1, and for relinquishing control of a unit of r_2 at s_2 at t_2. Typically, the mass will flow along a history whose graph connects (r_1,s_1,t_1) and (r_2,s_2,t_2), so that the "same" object is returned; but this formulation is somewhat more general (e.g. ~~for example~~, one may borrow a cup of sugar one day, and return a — presumably different — cup of sugar the next day). By allowing π to take on negative values, bailments as well as rentals may be encompassed.

Function π has the dimensions "dollars per unit mass", just as prices in general do. ~~One~~ We avoid~~s~~ complications by taking these to be discounted dollars. (Otherwise, ~~for example~~ e.g., ~~one has to~~ we must worry about whether the rental is to be paid at the beginning or the end of the period, or in periodic payments).

If one abstracts from legal complexities, tax liabilities, credit considerations, control restrictions, market frictions and imperfections, etc., a rental transaction may be thought of as a combination of two sale transactions. The agent acquiring temporary control in effect buys the object at the beginning of the rental interval and sells it back at the end of the period. In fact, if all the appropriate markets exist and the just-mentioned complications do not occur, one can give an informal argument for the following equality. For $t_1 < t_2$,

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$$\pi(r_1, s_1, t_1, r_2, s_2, t_2) = p(r_1, s_1, t_1) - p(r_2, s_2, t_2), \qquad (6.2.1)\ \cancel{(1)}$$

all prices being measured in discounted dollars. For if the left side were larger than the right, one could buy a unit of r_1 at (s_1, t_1), immediately rent it out, receive back a unit of r_2 at (s_2, t_2), immediately resell it, and emerge with a positive profit. If the left side were smaller than the right, one could acquire control over a unit of r_1 at (s_1, t_1), immediately sell it, then buy a unit of r_2 at (s_2, t_2) and hand it over to complete the rental transaction, again making a positive profit. With perfect information, arbitrage assures that neither of these inequalities obtains.

(A)

6.3. Imperfect Markets

~~Up to~~ Until now we have been making the following competitive assumptions: (i) The agent is faced with a price system $p: E_o \to$ reals ~~which~~ that does not depend on how much he buys or sells, and (ii) ~~There~~ there is a unique discount rate $k: T \to$ reals ~~which~~ that does not depend on the creditor-debtor position of the agent. We now briefly discuss weakening one or both of these conditions.

Let us first abandon condition (i), while keeping the perfect capital market assumption (ii). Let μ_1 and μ_2 be ~~net~~ sales and purchases (in physical, not value, terms). It will be convenient to consider net sales, $\mu = \mu_1 - \mu_2$; where μ is ~~of course~~ a signed measure, assumed bounded. Under ~~the~~ competitive assumption (i), the net revenue obtained from μ will be

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$$f(\mu) = \int_{E_o} p \, d\mu. \qquad (6.3.1) \; ~~(1)~~$$

Here p is assumed to be bounded measurable. Both p and net revenue are measured in discounted dollars.

The function f defined by (1) is linear; i.e. ~~That is~~, $f(\mu' + \mu'') = f(\mu') + f(\mu'')$, and $f(c\mu) = cf(\mu)$, for any two bounded signed measures μ', μ'', and any real number c.[6] But for imperfect markets, net revenue will in general be a non-linear function of net sales, and the problem arises of how to represent such functions in a convenient and plausible way.

Of the many possibilities, we shall consider here only representation by <u>densities</u>. This is a natural generalization of (1) ~~which~~ that appears to correspond quite well with the generalization from perfect to imperfect markets.

We define things abstractly. Let $(\underline{A},\Sigma)$ be a measurable space, and $\mathbb{M}$ be the set of all ~~sigma~~ σ-finite signed measures on it. Let $\underline{g}:\underline{A} \times \text{reals} \to \text{reals}$ be bounded measurable, and α be a fixed bounded measure on $(\underline{A},\Sigma)$. In terms of $\underline{g}$ and α, we define the function $\underline{f}:\mathbb{M}' \to \text{reals}$ by

$$\underline{f}(\mu) = \int_{\underline{A}} \underline{g}(\underline{a},\delta(\underline{a}))\,\alpha(d\underline{a}). \qquad (6.3.2)\ \cancel{(2)}$$

Here $\mathbb{M}'$ is the subset of $\mathbb{M}$ consisting of those signed measures ~~which~~ that are <u>absolutely</u> <u>continuous</u> with respect to α, and δ is the density of μ with respect to α:

$$\mu = \int \delta\, d\alpha. \qquad (6.3.3)\ \cancel{(3)}$$

Density δ exists by the Radon-Nikodym theorem. It is not unique, but any two densities for the same μ differ at most on a set of α-measure zero; hence they give the same value in (2), so that $\underline{f}$ is well-defined.

To see the connection between (1) and (2), consider the function $\underline{g}(\underline{a},\delta) = \underline{p}(\underline{a})\cdot\delta$. By (3), (2) then reduces to (1) (with $\underline{A}$ in place of $\underline{E}_o$).[7] But $\underline{g}$ in (2) can also be non-linear in δ, which leads to a non-linear $\underline{f}$.

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The interpretation of (2) is as follows: $g(a,\cdot)$ is the function giving net revenue density in terms of net sales density, at the point $a \in A$. Thus, for $\delta(a) > 0$, $g(a,\delta(a))/\delta(a)$ is the demand curve at a, and for $\delta(a) < 0$, it is the supply curve. (In both cases, quantity is the independent variable, and price is the dependent, variable.) All densities are with respect to α.

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The interpretation of α depends on the space A. Suppose ~~first that~~ we are dealing with just a single commodity, and that A is a bounded region of physical space, ~~S~~. (Net sales and net revenue may be thought of as steady flows per unit time). Then the natural interpretation for α is areal measure, and $g(a,\cdot)$ gives revenue in dollars per acre (per year) in terms of sales in, say, tons per acre (per year). For some problems this may be too restrictive. Suppose, for example, there are cities – (represented as geometric points) – at which one can garner positive revenue. Ordinary areal measure assigns measure zero to single points, which precludes representation in the form (2). This is easily remedied. To areal measure *per se*, we add a measure assigning unit mass to any point at which a city exists, and let this sum be α. Then representation by (2) is again possible; if there is a city at point a_o, $g(a_o,\cdot)$ gives net revenue per year accruing at a_o in terms of sales per year at that point.

Next, let A be a bounded subset of Space-Time, $S \times T$; we are again dealing with a single commodity. There is again a "natural" interpretation for α, viz. ~~namely~~, as the product measure formed from areal measure on S and Lebesgue measure on T. ~~And,~~ just as above, if markets are concentrated so that positive revenue accrues on a set of product measure zero, α may be modified to make representation in the form (2) possible.

Finally, we come to the case we started with, where $A = E_o \subseteq (R \times S \times T)$. In this case there does not seem to be any "natural" measure α, because there is nothing for R that corresponds to area for *Space* and "quantity of time" (Lebesgue measure) for *Time*. This creates no difficulty if there are just a finite number of resource-types, or even a countable number, say $\{r_1, r_2, \ldots\}$; for in this case ~~one~~ we choose~~s~~ a measure assigning a positive mass to each $\{r_n\}$ – say 2^{-n} – and then take~~s~~ the product of this with the *Space* and *Time* measures. And even in the general case there may be an α for which representation (2) is plausible.

Note that (2) has the same form as the utility function of the allocation-of-effort problem, chapter 5. Thus the problem of maximizing total net revenue in an imperfect market system, with a given endowment of goods, is encompassed in the results of that chapter.

However, (2) does have one rather important shortcoming as a representation of imperfect markets. While it allows variable prices, each price depends only on the quantity forthcoming in its own market (all "cross-elasticities" equal zero). In reality, ~~one~~ we would expect that a greater sale in a market would depress price not only there, but in markets for similar resource types at nearby space-time points. We ~~shall~~ do not go into the problem of representing this phenomenon.

We now turn to imperfections in the capital market. It will still be assumed that there is just one type of financial asset, "bonds," so that the model remains highly simplified. But the discount rate k will now be "personalized" and will depend on the creditor position of the agent in question (as well as on the time). A plausible ~~way to do this~~ method is to let k depend on b/v, the ratio of net bondholding to the value of physical assets of the agent. Thus the personal discount rate at time t will be

$$k(t,\ b(t)/v(t)) \qquad (6.3.4)$$

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~~where~~ k ~~will be~~ is a decreasing (or at least nonincreasing) function of its second argument. This reflects the following real-world situation. First, take the case of a creditor ($b > 0$, hence $b/v > 0$). As he extends more and more credit, the investment opportunities become progressively less attractive, so that k, the average rate of return, declines. Next consider a debtor (b and $b/v < 0$). For small debts one can rely on relatives and friends. Then one might try commercial banks. After ~~one's~~ the line of credit is exhausted here, one might try the "friendly finance" companies. And if even this does not suffice, there are always loan sharks and racketeers (who have a comparative advantage in enforcing collection of debt). At each stage ~~one's~~ a person's credit rating becomes shakier, and lenders compensate by charging higher interest rates. Thus k rises as b/v becomes more negative, and it is therefore again a decreasing function of its second argument.

The collateral constraint (1.11) ~~(11) of section 1,~~ may be interpreted in terms of (4). Suppose there is some negative value of b/v at which k goes to $+\infty$, which is to say that no more credit is forthcoming from any source beyond this point. ~~One~~ We then get~~s~~ a lower bound constraint on b/v, which in (1.11) ~~(11)~~ is equal to $-c/(1+c)$.

With (4), one can no longer speak unambiguously of "discounted dollars", because the size of discount itself depends on the agent's actions. The basic differential equation (1.1) ~~(1) of section 1,~~ connecting bondholding and sales becomes

$$Db(t) = k\big(t,\ b(t)/v(t)\big)b(t) + f(t), \qquad (6.3.5)\ \cancel{(5)}$$

where $f(t)$ ~~being~~ is the net rate (in current dollars) at which physical assets are being sold at time t (*net* rate means sales minus purchases). In general, (5) will no longer be integrable in elementary form.

Then (5), combined with given initial condition $b(t_o)$, $v(t_o)$, and either constraints (1.7) ~~(7)~~ or (1.10)–(1.11) ~~(10)-(11) of section 1,~~ then yields a system of conditions ~~which~~ that indicate what triples of time-paths $(f(t), b(t), v(t))$ are feasible.

Finally, let us combine imperfections in the commodity and capital markets. One additional problem now arises. Prices and revenues in commodity markets can no longer be expressed in discounted dollars; instead, we express them in current dollars.[8]

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We ~~have to~~ must complete our system of conditions by expressing the current dollar net rate of sales, $f(t)$ in (5), in terms of the signed measure of net sales, μ. (As above, the universe set of μ is $E_o \subseteq (R \times S \times T)$, the set of triples (r,s,t) for which markets exist; μ is measured in mass units). A measure α on E_o is needed to express net revenue in the form (2). For this, we postulate that a measure β on $(R \times S, \Sigma_r \times \Sigma_s)$ has already been arrived at, by some such process as discussed above. The product measure of β on $R \times S$ and Lebesgue measure on T, restricted to E_o, will be taken for α.

Signed measures μ ~~which~~ that are not absolutely continuous with respect to α are dismissed at once as infeasible. For any other μ, we form the density function $\delta = d\mu/d\alpha$, and substitute in (2); ~~(2)~~ which can be expressed as an iterated integral, first with respect to β over $R \times S$, then with respect to Lebesgue measure over T. The first iteration is the one that yields the net rate of sales function:

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$$f(t) = \int_{\{(r,s)\,|\,(r,s,t) \in E_o\}} g\big(r,s,t,\ \delta(r,s,t)\big)\,\beta(dr,ds)\,. \qquad \text{(6.3.6)} \quad \cancel{(6)}$$

subscript to S

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Here $\underline{g}$ is net revenue density, per unit time per "unit β". Performing the integration, we get the net revenue per unit time, which is $\underline{f}(\underline{t})$. Then (6) substituted in (5) ~~then~~ yields the basic relation between net bondholding $\underline{b}(\underline{t})$ and net sales μ. This, ~~together~~ with the other conditions mentioned after (5), then gives the set of feasible triples (μ, $\underline{b}(\underline{t})$, $\underline{v}(\underline{t})$).

A

6.4. The Real-Estate Market

The real-estate market distributes the control and ownership of Space-Time among economic agents.

That is, while a typical commodity market is characterized by a triple ($\underline{r}$,$\underline{s}$,$\underline{t}$), we are now dealing with pairs ($\underline{s}$,$\underline{t}$). The real-estate market is then the ensemble of all these separate point markets. The "homogeneity" of $\underline{S}$ and $\underline{T}$, as opposed to the "heterogeneity" of $\underline{R}$, makes the structure of the market here a good deal simpler than in the commodity case.

There are a number of different interpretations as to what exactly is being sold or rented in these markets. "Real estate" — or "land" — refers ambiguously to the Earth itself, with its soil, forests, waters, minerals, air, etc.; to the products of human construction ~~which are~~ more or less permanently affixed to it, such as buildings, roads, bridges, ~~etc.~~; or to the Space-Time continuum ~~which~~ that they occupy.

One can distinguish conceptually between control of these various components. Control over a portion of Space-Time is the right to exclude trespass by other agents, with their properties and associated activities. This is not quite the same as control over the use to which a building in the region may be put. In practice, of course, it would be inconvenient to have these rights in the hands of agents with opposing wills. Hence control is generally vested in one agent, although there may be separate markets for the components of a real-estate package, and ownership may be scattered among several agents.

For most of our analysis it does not matter which of the various possible interpretations is used: whether the real-estate transaction is just for the Space-Time "shell", or whether the constructed or natural contents of the region are part of the package. We ~~shall~~ ignore the ambiguity whenever the analysis applies to any of the interpretations.

The real-estate market has two further simplifying peculiarities as compared with commodity markets. First, the "amount" of Space-Time anywhere is fixed. Second, the control or ownership of each point (s,t) is usually in the hands of ~~just~~ one agent. This suggests that we represent the control or ownership of any one agent as a (measurable) subset of $S \times T$, rather than as a measure over $S \times T$.

In more detail, we suppose that each agent i chooses a subset $E_i \in (\Sigma_s \times \Sigma_t)$. The chosen subsets for different agents are disjoint, so that the collection of all the E_i's is a packing. Thus E_i is that portion of Space-Time ~~which is~~ under the exclusive control (or ownership) of agent i. (Later we ~~shall~~ generalize to allow for joint ownership or control, but the model just given will serve for most of this section).

As an example, E_i might be of the form $F \times G$, where F is a region of Space and G is an interval of Time, say (t_1, t_2). This means that agent i acquires region F at moment t_1, keeps it until moment t_2, at which time he divests himself of it, and has no other portion of $S \times T$. More generally, E_i might be a union of such "rectangles", different regions being held for different time-intervals. Thus E_i might consist of pieces scattered all over the world, as might be the case if agent i is an international corporation. The size, shape, number, and duration of ~~one's~~ holdings are all determined by the set E_i.

9e ¶ ~~Let us~~ We next consider feasibility restrictions on the possible sets E that can be chosen. Some restrictions hold for all agents, while others hold for selected types of agents. We first consider universal restrictions.

A number of these arise from the need for informational economy. This leads to a restriction in the variety of possible transactions. Thus real-estate transactions are almost universally of the rectangular form mentioned above: One ~~acquires~~ a parcel F is acquired for a time interval G. (This includes the case of outright sale of a parcel. If the parcel is never resold, the time interval G extends to the infinite future.) There are usually further restrictions on F and G. For F in particular, it is typical to partition a portion of the Earth's surface into **lots**, with the stipulation that the lot must change hands as a unit. if F is a lot, and F_i is the region held by agent i, then either $F \subseteq F_i$, or $F \cap F_i = \emptyset$.

(As for the vertical dimension, one can imagine the lot F as actually representing a three-dimensional cone, with its apex at the center of the Earth, and projecting through F at the Earth's surface to infinity. But subsoil rights are sometimes transacted for separately, and it is not at all clear how high a person's air rights extend. Would someone have the right to build a structure on his property so tall that it interfered with airline flight paths?)

The available sets E would ~~then~~ be restricted to unions of ~~these~~ allowable rectangles.

A good portion of the Earth's surface is not available for transactions at any given time; e.g., unclaimed territory, the high seas, and the public domain, including the road and street system.

There may be maximal limits to the holdings of any one agent in certain regions, a result of land reform movements.

As for particularistic restrictions on landholdings, these have been applied historically to aliens and certain minority groups, such as the Jews in Russia or the Japanese in California. Private restrictive covenants will limit the market still further for certain groups.

We ~~shall~~ now go a step beyond our analysis of commodity markets — which stopped with a discussion of feasibility conditions — and investigate the full conditions of equilibrium in the real-estate market. This involves a discussion of the <u>preferences</u> of the market participants, and the conditions under which a pricing system clears the market. We ~~shall~~ end with a proof of the existence of equilibrium under certain simplifying assumptions.

Ideally ~~one~~ we want~~s~~ a model that simultaneously determines the pattern of ownership over Space-Time, the pattern of control, and the pattern of land uses, because these three systems are interrelated. An agent wants control of a certain region ~~in order~~ to operate certain activities there. Which activities are feasible depends on a number of factors. Among these are technical knowledge, budgetary limits, legal constraints -- such as housing and zoning laws -- activities in adjacent regions (in the case of neighborhood effects), and, in particular, the capital endowment resulting from previous land uses on the same site. The desirability of various regions to an agent is a reflection of the desirability of these possible uses to him.

As for ownership, there is -- (apart from any "psychic income" received from having a stake in the land) -- a comparative advantage from the agent's owning land that he controls, because the inevitable frictions and inefficiencies ~~which~~ that accompany the rental relationship are avoided. In the real world ~~there is~~ a very close association exists between the pattern of ownership and the pattern of control, which results from this phenomenon.

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The structure of equilibrium in the real-estate market needs further specification. We suppose ~~that~~ there is a perfect capital market, so that all prices and rentals may be measured in <u>discounted</u> dollars. Then we postulate that at equilibrium there will be a <u>rental measure</u>, μ, over a measurable subset E_o of $(S \times T, \Sigma_s \times \Sigma_t)$, with the interpretation: $\mu(E)$ = rental (in discounted dollars) for the control of Space-Time "region" $E \subseteq E_o$.

A number of comments follow. First, ~~one~~ we should distinguish carefully between regions <u>per se</u> and "regions" of Space-Time, which are measurable subsets of S, and $S \times T$, respectively. The context will make clear which we are talking about, and quotation marks will not be used.

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Second, it may not be possible to assign a rental value to all measurable subsets of E_o in an empirically meaningful way. If parcels are always transacted for as units, there is no rental value for a fraction of a parcel. This difficulty is easily remedied: ~~One~~ We simply ~~aggregates~~ aggregate μ to the appropriate sub-~~sigma~~ σ-field of $\Sigma_s \times \Sigma_t$. If E_o is partitioned into rectangles $F \times G$, F ranging over the minimal subdivision units and G over the minimal time intervals for which transactions occur, the appropriate ~~sigma~~ σ-field is the one generated by this partition. Since no difficulties arise, we suppose this aggregation has been done, but retain our original notation.

The last comment is more substantive. By postulating μ, we have willfully fallen into the trap we warned against in the discussion of rental markets. The rental for region E should in principle depend on the activities ~~which the~~ tenant operates in E. If he returns it with a dilapidated capital endowment, a higher rental would presumably be charged in compensation. If he returns it in improved condition, the rent would be lower, possibly negative (as when an owner turns his land over to a developer). The rental measure μ, however, implies that rent does not depend on land use.

This is done mainly for simplicity's sake, but ~~one~~ we might justify it as an approximation under certain special conditions. One condition is that the activities contemplated have no "construction" or "mining" components, so that alternative activities would have little differential effect on capital endowment. Once the structures are in place, alternative office activities, ~~or~~ manufacturing activities, or residential living activities probably do not make much difference in depreciation rates. The situation is different with farming, fishing and forestry, as well as mining and construction *per se*.

A second condition (this is more dubious) under which the effects of activities on rentals may conceivably be ignored is when the market is for Space-Time *per se* as separate from its contents. The rental is then "ground rent" only, and is presumably affected largely by the overall "location" of the region in $S \times T$, and relatively little by the particular site characteristics such as the capital endowment.

We are also implicitly assuming away neighborhood effects, since otherwise rentals would be affected by the activities operating in adjacent regions.

In any case, we postulate μ. μ and all other measures ~~in~~ this section are assumed to be finite. (A generalization will be discussed later). Let us now go on to land values. For each time t, we suppose ~~that~~ there is an equilibrium land-value measure, μ_t, whose universe set, F_t, is a region of S, with the interpretation: $\mu_t(F)$ is the sale value (in discounted dollars) of region $F \subseteq F_t$.

Note that the land-value measures μ_t are over subsets of Space, while the rental measure μ is over a subset of Space-Time. An argument similar to that for (2.1) ~~(1) of section 2~~ suggests the relation, for $t_1 < t_2$,

$$\mu\Big(F \times \{t \mid t_1 \leq t < t_2\}\Big) = \mu_{t_1}(F) - \mu_{t_2}(F), \qquad (6.4.1)\ \cancel{(1)}$$

for any region $F \subseteq S$ for which all these markets exist. Equation (1) states that the value of a parcel at t_1 equals what you can get by selling it later at t_2, plus the rental obtained for the interval, when everything is measured in discounted dollars. This is not unreasonable, but it does assume away market frictions, ignorance, psychic income from ownership, etc.

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An immediate consequence of (1) is the following. Let Space-Time region E be a disjoint union of n rectangles:

$F_j \times \{t | t_{j1} \le t < t_{j2}\}$, $j = 1,\ldots,n$. One can attain ownership of just E by 2n transactions, buying at t_{j1}, and selling at t_{j2}. (1) then implies that the net expenditure for the ownership of E is exactly the same as the rental charge for E.

We now discuss the concrete organization of the market. Everything starts at time t_o, and there is an initial distribution of land ownership. Agent i owns region F_i, the collection of all the F_i's being a packing in Space. The real-estate market then operates to create two measurable partitions of E_o: a partition by ownership, and a partition by control — say E'_i is ~~that~~ the portion of Space-Time ~~which~~ that comes to be owned by agent i, and E''_i is the ~~that~~ portion ~~which~~ that comes to be controlled.

Here E_o is the region of $S \times T$ on which transactions can occur. None of E_o is assumed to occur prior to time t_o. We ~~shall~~ also suppose that each ownership region E'_i is a finite union of disjoint rectangles, as just discussed. The number of participants in the market will be assumed finite, except in one or two discussions below.

6.5. Real-Estate Preferences

Now we come to preferences. It is assumed that each agent has a preference ordering over triples (E', E'', x), where E' and E'' are measurable subsets of E_o (E' in the union of rectangles form) and x is a real number. This triple represents the situation in which the agent owns region E', controls region E'', and has a net expenditure of x on real-estate transactions (in discounted dollars).

This is a somewhat unusual set of objects over which to express preferences, but it is just the right thing for capturing the options available to the participant in the real-estate market. Note that budgetary stringencies may be reflected in this preference order: A poverty-stricken agent, or one sailing close to the wind, will give relatively heavy weight to variations in x.

~~Supposing~~ Given the land-value and rental measures, ~~given, just~~ what *will* be the net expenditures of agent i if he chooses E' to own and E'' to control? The answer is

$$x = \mu(E'') - \mu_{t_o}(F_i), \qquad (6.5.1)\ ~~(1)~~$$

~~that is~~ i.e., the rental for his control set E'', net of the value of his initial holding F_i. Relation (1) seems rather surprising, since it is independent of E'. To demonstrate (1), we think of the agent as making four transactions: (i) selling his initial holding; (ii) buying E'; (iii) renting out $E' \setminus E''$ (the portion of E' he does not choose to control); and (iv) acquiring control over $E'' \setminus E'$ (which, together with $E'' \cap E'$, gives him the set E'' for control). The net expenditures for these four transactions are the four respective terms in

$$-\mu_{t_o}(F_i) + \mu(E') - \mu(E' \setminus E'') + \mu(E'' \setminus E'), \qquad (6.5.2)\ ~~(2)~~$$

which is the same as (1). (If the "bases" of some of the rectangles constituting E' overlap F_i at time t_o, ~~then~~ a part of (i) and (ii) is a fictitious transaction in which the agent sells to himself. This, of course, will have no effect on the sum (2)).

Intuitively, $\underline{E}'$ does not enter (1) because it is bought and then rented ~~out~~ (partly to the agent himself perhaps), hence it incurs a net expenditure of zero, by ~~(1)~~ (4.1) ~~of the preceding section~~.
This underlines the assumptions behind that equation, which can be put roughly as follows. The pattern of ownership really doesn't matter, because markets operate without friction and nobody gets any psychic income from owning land *per se*.

We now specialize our assumptions concerning preferences to bring them into line with this approach; ~~namely~~ viz., we assume that each agent is indifferent to variations in the first term of the triple $(\underline{E}', \underline{E}'', \underline{x})$. This is a considerable simplification, because it means that each agent has a well-defined preference order over pairs $(\underline{E}'', \underline{x})$. Assuming (for simplicity of notation if nothing else) that these orderings may be represented by utility functions, we have for each agent $\underline{i}$ a function

$$U_i(E, x) \qquad (6.5.3) \; \cancel{(3)}$$

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giving his preferences over combinations of $\underline{E}$, the region of Space-Time that he controls, and $\underline{x}$, ~~his~~ the net discounted expenditures on real-estate. Region $\underline{E} = \underline{E}''$ and $\underline{x}$ are connected by relation (1).

The conditions for equilibrium in the market for control of Space-Time may now be stated. Given initial holdings $\underline{F}_i$

at time t_o, and utility functions (3) for all agents, the equilibrium consists of (i) a bounded rental measure, μ, on E_o, (ii) a bounded land-value measure μ_{t_o} for the initial time t_o, and (iii) a measurable partition (E_i) of E_o among the agents i; and

$$U_i\left[E,\ \mu(E) - \mu_{t_o}(F_i)\right] \qquad \text{12 pt brackets} \qquad (6.5.4)\ \cancel{(4)}$$

must be maximized at $E = E_i$ over all possible measurable subsets E of E_o, for each agent i.

That is, given the relevant prices for regions, no agent can choose a more preferred region than the one he actually has chosen, and these chosen regions partition the market.

Note that the market for ownership of real estate has dropped out of sight, except for the initial holdings. We shall in fact concentrate all our attention on the control market.

Even this simplified model seems still ~~to~~ too general to give interesting results. We therefore consider a further specialization of (3), with the utility function U_i in the form

$$U_i(E,\ x) = V_i(E) - x \qquad (6.5.5)\ \cancel{(5)}$$

for all agents i. Function (5) represents the assumption of "constant marginal utility of money", and may be taken as a reasonable approximation ~~in the case~~ when real-estate expenditures are just a small fraction of one's total expenditures.

Function (5) leads to a great simplification in the conditions of equilibrium. Specializing (4), we find that the conditions for μ and the partition (E_i) are independent of the initial holdings F_i. The conditions are that E_i must maximize

$$V_i(E) - \mu(E) \qquad (6.5.6) \quad ~~(6)~~$$

over all measurable $E \subseteq E_o$, for each agent i.

In the special utility function (5), V_i is a set function, whose domain is the ~~sigma~~ σ-field on E_o. If, in particular, V_i is a bounded signed measure for each i, ~~then~~ we can and will prove the existence of an equilibrium for the real-estate market. ~~We shall do this presently,~~ but first let us contemplate the stated condition on V_i and discuss its plausibility.

First, V_i is allowed to be a signed measure, so that it might conceivably take on negative values for certain sets. This means that agent i would prefer not to have control over certain regions. Is this realistic? Control, in fact, is typically attended with some obligations or other disabilities, and these might occasionally outweigh the benefits of control. Examples are legal liability for accidents, ~~for~~ maintaining nuisances, and ~~for~~ paying property taxes (in the cases where these liabilities devolve on the tenant rather than the land-lord), the onus of being a "slumlord", etc.

Thus, since it might have some applications, and ~~since it~~ creates no mathematical complications, we ~~shall~~ keep the generality of using signed measures rather than measures per se.

Note, however, that the equilibrium rental distribution might in this case also turn out to be a (proper) signed measure.

More interesting is the additivity condition on V_i. Letting E and F be two disjoint regions, is it plausible that

(6.57)

$$V_i(E \cup F) = V_i(E) + V_i(F)? \qquad (\cancel{6}\ 7)$$

This says, roughly, that the desirability of controlling a region does not depend on what other regions agent i controls.

In general, (~~6~~ 7) will not hold in the real world. ~~In fact,~~ one can think of several situations where the left side of (~~6~~ 7) should be smaller than the right, and several others where it should be larger. The "smaller" case arises because regions can be substitutes for each other. Let E and F be alternative plots suitable for residential use by person i. He might have little need for both ~~of them~~, and therefore be hardly willing to pay more for E and F combined than for either ~~one~~ alone. ~~And,~~ in general, after a certain amount of land at the right places and times has been acquired, an agent will not be able to make much use of additional land.

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Conversely, the left side of (~~6~~ 7) will exceed the right when the regions E and F are complements. They might be too small separately to accommodate a certain projected land use, but adequate together. In this case, agent i might be unwilling to pay much for one of these regions without the other, but a good deal for both together. "Too small" can refer either to Space or Time or both. Thus, suppose E and F are both rectangles in $S \times T$. They may be adjacent parcels over the same time interval, together adequate to accommodate a plant (or a complex of "linked" plants) of efficient size, but too small separately. They may be successive time intervals of the same parcel, each too short alone for a certain developmental project but adequate together. In this case agent _i_ might be willing to pay a premium to have a long-term lease encompassing both _E_ and _F_.

There are several real-world manifestations of this effect. Plants buy up excess land in hopes of inducing "linked" plants to settle there.[9] Large parcels tend to be worth more per square foot than small ones.[10] These phenomena would be hard to explain if ~~(6)~~ (7) held.

In general, the _shape_ of ~~one's~~ the region of control is an important factor in its value to an agent. Connected, "chunky" parcels, if small, tend to be preferred to fragmented or elongated ones, ~~The reason is that~~ because the uses planned by one agent will generally have heavy transport-communication flows among themselves: the movements of the agent himself, the flow of goods in an integrated plant system, the flow of messages between headquarters and field offices~~, etc.~~ A "tight" site pattern tends to save on these "connection" costs.

~~There are~~ Certain exceptions, ~~which~~ however, prove the rule that certain shapes facilitate interaction better than others. A multistage assembly line process (as in automobile manufacture) might be best housed in a long narrow plant site. A similar argument has been applied to agriculture, in terms of ease of plowing and communication with the world at large.[11]

These examples are spatial, but similar arguments apply to ~~Time~~ time. For the same total duration, one connected stretch is generally preferred to a number of small interrupted intervals.

These arguments apply mainly to "relatively small" regions. For "large" regions, the substitutability among neighboring points begins to outweigh the complementarity, and one prefers to have ~~one's~~ sites scattered; ~~For example~~ e.g., chain stores spread out rather than agglomerate.

None of these "shape" effects would arise if the additivity condition (7) were valid. Nonetheless, we persist in assuming it, as a mathematically tractable first approximation, ~~and one~~ which compromises between the two opposite tendencies ~~we have~~ just discussed.

6.6. Equilibrium in the Real-Estate Market

We ~~shall~~ state the problem abstractly. Given measurable space $(\underline{A},\Sigma)$, and $\underline{n}$ bounded signed measures, $\mu_1,\ldots,\mu_n$, do ~~there exist~~ $\underline{n}$ measurable subsets, $E_1^o,\ldots,E_n^o$, and a bounded signed measure μ^o, exist such that

(i) the collection $\{E_1^o,\ldots,E_n^o\}$ is a partition of $\underline{A}$, and

(ii) for each $\underline{i} = 1,\ldots,n$, E_i^o maximizes

$$\mu_i(\underline{E}) - \mu^o(\underline{E}) \qquad (6.6.1)\ \cancel{(1)}$$

over all measurable subsets $\underline{E}$?

The interpretation we have in mind is that $\underline{A}$ is the subset of $\underline{S} \times \underline{T}$ for which the real-estate market exists; μ^{o} is the rental (signed) measure; there are $\underline{n}$ participants in the market, and E_i^o is the region chosen by agent $\underline{i}$ for his control, the net cost to him being $\mu^{o}(E_i^o)$; (1) is the same as ~~(6) of section 5~~ (5.6) and is the utility level of agent $\underline{i}$ if he acquires region $\underline{E}$. (We have changed the notation V_i to μ_i, since we are assuming this function is a (signed) measure). Conditions (i) and (ii) are then precisely the equilibrium conditions for the real-estate market.

E_i^o

We shall prove the existence of equilibrium. Note that the number of participants in the market is finite, and ~~also that~~ all the μ_i's are also finite. The first condition is essential; the second can be removed, but (1) then requires some reinterpretation. This will be done in ~~a later~~ section §6.9.

Our method of procedure will be indirect, and we shall prove several other properties of the equilibrium. These~~, in fact,~~ have independent economic meanings, and are of interest in themselves. Also, this procedure paves the way for the generalization to pseudomeasures ~~which~~ that comes later.

§9.5

Recall that a Hahn decomposition of a signed measure μ is a pair of measurable sets, $(\underline{P},\underline{N})$, ~~which~~ that partition universe set $\underline{A}$, and ~~are~~ for which $\mu(\underline{E}) \geq 0$ on all measurable sets $\underline{E} \subseteq \underline{P}$, and $\mu(\underline{F}) \leq 0$, all measurable $\underline{F} \subseteq \underline{N}$. We now need a generalization of this concept to several signed measures.

#

¶ **Definition**: Given measurable space (A, Σ) and signed measures $(\mu_1, \ldots, \mu_n)$, an extended Hahn decomposition is an n-tuple of measurable sets $(E_1, \ldots, E_n)$ which partitions A and for which

$$\mu_i(F) \geq \mu_j(F), \qquad (6.6.2)\ \cancel{(2)}$$

on all measurable $F \subseteq E_i$, ~~all~~ $i, j = 1, \ldots, n$.

That is, on E_i, μ_i is at least as large as any of the other signed measures μ_j $(j = 1, \ldots, n)$.

The economic interpretation of the condition that $(E_1^o, \ldots, E_n^o)$ is an extended Hahn decomposition is that, on any measurable subset of E_i^o, agent i will at least match the bid of any other agent. This appears to be a reasonable alternative definition for a partition being an equilibrium for the real-estate market.

This definition of equilibrium looks quite different from the definition given by (1). For one thing, it involves no mention of any rental distribution μ^o. But our next result shows that, in fact, ~~in fact,~~ these two definitions are equivalent.

¶ **Theorem**: Let μ^o and $\mu_1, \ldots, \mu_n$ be bounded signed measures on measurable space (A, Σ), and let $(E_1^o, \ldots, E_n^o)$ be measurable sets ~~which~~ that partition A. Then μ^o satisfies one of the following two conditions iff it satisfies the other:

(i) for each $i = 1, \ldots, n$, E_i^o maximizes

$$\mu_i(E) - \mu^o(E) \qquad (6.6.3)\ \cancel{(3)}$$

P (ii) align right

over all $E \in \Sigma$;

(ii) for all i, $j = 1,\ldots,n$ $(i \neq j)$,

$$\mu_i(F) \geq \mu^\circ(F) \geq \mu_j(F), \qquad \text{(6.6.4)} \; \cancel{(4)}$$

for any measurable $F \subseteq E_i^\circ$.

Furthermore, ~~there exists~~ a bounded signed μ° exists satisfying one (hence both) of these conditions iff $(E_1^\circ,\ldots,E_n^\circ)$ is an extended Hahn decomposition for $(\mu_1,\ldots,\mu_n)$.

Proof: Let E_i° maximize (3), for all i, and take measurable $F \subseteq E_i^\circ$. Then

$$\mu_i(E_i^\circ) - \mu^\circ(E_i^\circ) \geq \mu_i(E_i^\circ \backslash F) - \mu^\circ(E_i^\circ \backslash F).$$

Simplification yields the left inequality in (4). Also,

$$\mu_j(E_j^\circ) - \mu^\circ(E_j^\circ) \geq \mu_j(E_j^\circ \cup F) - \mu^\circ(E_j^\circ \cup F).$$

Simplification yields the right inequality in (4).

Conversely, let (4) hold, and take any $E \in \Sigma$. Then (3) is the sum of n terms: $\mu_i(E \cap E_j^\circ) - \mu^\circ(E \cap E_j^\circ)$, $j = 1,\ldots,n$. For all $j \neq i$, these terms are ≤ 0, by the right inequality in (4). Hence

$$\mu_i(E) - \mu^\circ(E) \leq \mu_i(E \cap E_i^\circ) - \mu^\circ(E \cap E_i^\circ). \qquad \text{(6.6.5)} \; \cancel{(5)}$$

Also,

$$0 \leq \mu_i(E_i^\circ \backslash E) - \mu^\circ(E_i^\circ \backslash E), \qquad \text{(6.6.6)} \; \cancel{(6)}$$

by the left inequality of (4). Adding (5) and (6), we obtain

break at ≤

$$\underline{\mu}_i(\underline{E}) - \underline{\mu}^o(\underline{E}) \leq \underline{\mu}_i(\underline{E}^o_i) - \underline{\mu}^o(\underline{E}^o_i),$$

$1\frac{1}{4}5$

so that $\underline{E}^o_i$ does indeed maximize (3). This proves the equivalence of conditions (i) and (ii).

A-6

Next, suppose there is a μ^o satisfying these conditions. The extended Hahn decomposition condition (2) follows at once from (4).

Finally, suppose $(\underline{E}^o_1,\ldots,\underline{E}^o_n)$ is an extended Hahn decomposition for $(\mu_1,\ldots\mu_n)$. For μ^o choose the signed measure given by

$$\mu^o(\underline{E}) = \mu_1(\underline{E} \cap \underline{E}^o_1) + \ldots + \mu_n(\underline{E} \cap \underline{E}^o_n), \qquad \text{(6.6.7)} \; \cancel{(7)}$$

all $\underline{E} \in \Sigma$. (~~That is~~ i.e., μ^o is the direct sum or patching of the μ_i respectively restricted to $\underline{E}^o_i$, $\underline{i} = 1,\ldots,n$). Then for measurable $\underline{F} \subseteq \underline{E}_i$, we obtain

break at =

$$\mu^o(\underline{F}) = \mu_{\underline{i}}(\underline{F}) \geq \mu_j(\underline{F}),$$

from (7) and (2). This implies (4). ||| $\square$

A Condition (3) is the same as (1), so that (i) is precisely our original condition that μ^o and $(\underline{E}^o_1,\ldots,\underline{E}^o_n)$ should be an equilibrium for the real-estate market. We have just shown that there is a μ^o such that this is the case iff $(\underline{E}^o_1,\ldots,\underline{E}^o_n)$ is an extended Hahn decomposition. Thus in this sense the two equilibrium concepts coincide. In addition, the result just obtained states a necessary and sufficient condition for a given signed measure μ° to be the rental distribution in the real-estate equilibrium. This is (4), and it, too, has a simple economic interpretation: ~~namely, that~~ the price at which something sells must lie between the highest and second-highest offers of the bidders in the market.

We ~~shall~~ now prove the existence of an equilibrium. ~~This will be done~~ by proving that an extended Hahn decomposition, $(E_1^{\circ},\ldots,E_n^{\circ})$, exists. The preceding theorem then implies the existence of a rental signed measure μ°, and together these satisfy the equilibrium conditions (1) or (3).

In fact, we ~~shall~~ prove something stronger: Given signed measures $\mu_1,\ldots,\mu_n$ (not necessarily finite, or even ~~sigma~~-finite), if each pair of these has an extended Hahn decomposition, then so does the whole n-tuple $(\mu_1,\ldots,\mu_n)$. To see that this implies the existence of a decomposition in the case we are considering, let μ_i and μ_j be bounded signed measures. Then $\mu_i - \mu_j$ is a signed measure. Hence, by the ordinary Hahn decomposition theorem, there is a pair of measurable sets (P,N), that ~~which~~ partitions A, and for which

break at - only

$$\mu_i(F) - \mu_j(F) \; (\geq, \leq) \; 0,$$

for measurable $F \subseteq P$, $F \subseteq N$, respectively. But this is precisely the condition (2) for (P,N) to be an extended Hahn decomposition for the pair (μ_i, μ_j). Hence such a decomposition exists for any pair of bounded signed measures. The premise of the following theorem is therefore fulfilled, and we conclude that a decomposition exists for any *n*-tuple of bounded signed measures.

#

D

Theorem: Let $(\mu_1,\ldots,\mu_n)$ be an *n*-tuple of signed measures on space (A,Σ), such that each pair of these has an extended Hahn decomposition. Then the whole *n*-tuple has an extended Hahn decomposition.

#

D

Proofs

Proof: By induction on *n*. The statement is true for $n = 2$ by assumption. For $n > 2$, ~~let us~~ we suppose it holds for $n - 1$, and prove it for *n*.

Thus, for $(\mu_1,\ldots,\mu_{n-1})$ there is a measurable $(n-1)$-tuple $(E_1,\ldots,E_{n-1})$ ~~which~~ that partitions A, and for which

$$\mu_i(F) \geq \mu_j(F), \qquad (6.6.8)\ \cancel{(8)}$$

for all measurable $F \subseteq E_i$, ~~all~~ $i, j = 1,\ldots,n-1$.

For each $i = 1,\ldots,n-1$, there is by assumption an extended Hahn decomposition (P_i, N_i) for the pair (μ_i, μ_n). Thus

$$\mu_i(F) \geq \mu_n(F) \qquad (6.6.9)\ \cancel{(9)}$$

g.19-7

for measurable $F \subseteq P_i$, and, for $F \subseteq N_i$, the opposite inequality holds. Now define $(E_1^o,\ldots,E_n^o)$ by

$$E_i^o = E_i \cap P_i \qquad (6.6.10)\ \cancel{(10)}$$

We have ~~now~~ shown that $(E_1^\circ, \ldots, E_n^\circ)$ given by (10) and (11) is an extended Hahn decomposition for $(\mu_1, \ldots, \mu_n)$. This completes the induction and the proof. $\square$

We have now proved the existence of equilibrium in the real-estate market with a finite number of participants. What if the number of participants is infinite? (This might occur with an unbounded Space or Time horizon). ~~Now~~ there is no trouble extending the definition of equilibrium, and of extended Hahn decomposition, to the case of an infinite number of participants with corresponding bounded signed measures μ_i. This will involve an infinite measurable partition of A, the pieces in 1-1 correspondence with the μ_i's, and the relations (1) or (2) holding for each piece. In fact, we make this very extension later.

But is it true that this equilibrium, or decomposition, always exists? The answer is no, as the following trivial counterexample demonstrates. Let the space A consist of just one point. Let $\mu_1, \mu_2, \ldots$ be an infinite sequence of measures, ~~namely~~ viz., $\mu_n(A) = 1 - 1/n$, $n = 1,2,\ldots$. ~~It is obvious that~~ No equilibrium, and no extended Hahn decomposition, exist~~s~~, since no matter to what participant n we assign the single point of A, he is "outbid" by participant $n+1$.

Let us turn briefly to the question of uniqueness of equilibrium.

Theorem: Let $(\mu_1,\ldots,\mu_n)$ be an n-tuple of signed measures on space (A,Σ), with extended Hahn decomposition $(E_1^o,\ldots,E_n^o)$; let $(E_1,\ldots,E_n)$ be another n-tuple of measurable sets ~~which~~ that partitions A. Then $(E_1,\ldots,E_n)$ is also an extended Hahn decomposition iff

$$\mu_i(F) = \mu_j(F) \qquad (6.6.14)\ \cancel{(14)}$$

for all measurable $F \subseteq (E_i^o \cap E_j)$, ~~all~~ $i, j = 1,\ldots,n$.

Proof: Let $(E_1,\ldots,E_n)$ be another decomposition, and take measurable $F \subseteq (E_i^o \cap E_j)$. Then $\mu_i(F) \geq \mu_j(F) \geq \mu_i(F)$, from (2), which yields (14).

Conversely, let (14) hold, and take measurable $F \subseteq E_i$.

$$\begin{aligned}\mu_i(F) &= \mu_i(F \cap E_1^o)+\ldots+ \mu_i(F \cap E_n^o)\\ &= \mu_1(F \cap E_1^o)+\ldots+ \mu_n(F \cap E_n^o)\\ &\geq \mu_j(F \cap E_1^o)+\ldots+ \mu_j(F \cap E_n^o) = \mu_j(F),\end{aligned} \qquad (6.6.15)\ \cancel{(15)}$$

for any $j = 1,\ldots,n$. The first and last equalities in (15) arise from the fact that $\{E_1^o,\ldots,E_n^o\}$ is a partition; the middle equality arises from (14) and the fact that $(F \cap E_k^o) \subseteq (E_k^o \cap E_i)$; the inequality in (15) arises from the fact that $(E_1^o,\ldots,E_n^o)$ is an extended decomposition.

Then (15) implies that $\mu_i(F) \geq \mu_j(F)$ when $F \subseteq E_i$, so that $(E_1,\ldots,E_n)$ is indeed another decomposition. |||

This theorem has again a simple economic interpretation. A given region F can be under different controllers in two equilibria iff their bids over this region are identical. Condition (4) for the rental distribution μ° then implies that μ° is identical over F to this common signed measure. The two agents are then indifferent about controlling F, the rental just cancelling out the benefits they receive from this control.

Thus, while multiple equilibria are possible, they are of a somewhat trivial character from a practical point of view. The extreme case occurs when all the μ_i's are identical (all agents have identical preferences). One easily verifies that any measurable partition into n pieces is an extended Hahn decomposition in this case, and that it is an equilibrium; the rental distribution μ° is this common signed measure, and all agents are indifferent among all possible regions.

It is very common for equilibria to satisfy one or another extremal condition ~~which~~ that can be given a welfare interpretation of sorts. Our final result of this section is of this character.

D

Theorem: Let $(\mu_1,\ldots,\mu_n)$ be an n-tuple of bounded signed measures on space (A,Σ). The n-tuple of sets $(E_1^\circ,\ldots,E_n^\circ)$ is an extended Hahn decomposition iff it maximizes

$$\mu_1(E_1)+\ldots+\mu_n(E_n) \qquad \text{(6.6.16)} \; \cancel{(16)}$$

over all measurable n-tuples $(E_1,\ldots,E_n)$ ~~which~~ that partition A.

Proof: Let $(E_1^\circ,\ldots,E_n^\circ)$ be an extended Hahn decomposition, ~~and~~ a let $(E_1,\ldots,E_n)$ be another measurable n-tuple ~~which~~ that partitions A. Then

$$\mu_i(E_i^\circ \cap E_j) \geq \mu_j(E_i^\circ \cap E_j) \qquad (6.6.17)\ \cancel{(17)}$$

for all i, $j = 1,\ldots,n$, by (2). Adding the inequalities (17) over the n^2 possible (i, j) pairs yields

$$\mu_1(E_1^\circ)+\ldots+\mu_n(E_n^\circ) \geq \mu_1(E_1)+\ldots+\mu_n(E_n), \qquad (6.6.18)\ \cancel{(18)}$$

so that $(E_1^\circ,\ldots,E_n^\circ)$ does indeed maximize (16).

Conversely, let $(E_1^\circ,\ldots,E_n^\circ)$ maximize (16), and take measurable $F \subseteq E_i^\circ$. Define $E_i = E_i^\circ \setminus F$, $E_j = E_j^\circ \cup F$, and $E_k = E_k^\circ$ for all $k \neq i$, $k \neq j$, $k = 1,\ldots,n$. Then the n-tuple $(E_1,\ldots,E_n)$ partitions A. We then have (18), which simplifies to

$$\mu_i(E_i^\circ) + \mu_j(E_j^\circ) \geq \mu_i(E_i^\circ \setminus F) + \mu_j(E_j^\circ \cup F),$$

and this in turn simplifies to $\mu_i(F) \geq \mu_j(F)$. Thus $(E_1^\circ,\ldots,E_n^\circ)$ is an extended Hahn decomposition. ~~|||~~ $\square$

This again has a simple economic interpretation. If $(E_1,\ldots,E_n)$ represents the way ~~in which~~ Space-Time is partitioned among the n agents, ~~then~~ (16) may be thought of as a kind of social valuation of this partition. It is the sum of the values ~~which~~ each participant personally places on the region allotted to himself. (These values are all measured in the same units, ~~namely,~~ viz. discounted dollars).

g19-9 Thus a given partition is an equilibrium for the real-estate market iff it maximizes the social valuation of land in discounted dollars, as given by (16). This is a useful observation, though ~~one~~ cannot draw welfare or policy conclusions from it without further assumptions.

In all the preceding analysis it was required that the n-tuples $(E_1,\ldots,E_n)$ form a partition of the universe set A, rather than just a packing. ~~In other words,~~ the entire region had to be allocated; the possibility of leaving part of it vacant was not allowed. At first glance this seems rather restrictive, especially ~~in view of the fact that~~ since we allow signed measures, so that some agents would prefer not to control some regions. In the real world, would not a universally repugnant region be left uncontrolled?

But ~~in fact~~ this apparent generalization is already contained in the preceding model. We simply add a "dummy" participant $n+1$, with μ_{n+1} identically zero. Letting $(E_1^\circ,\ldots,E_{n+1}^\circ)$ be an extended Hahn decomposition for $(\mu_1,\ldots,\mu_{n+1})$, we find from (2) that all the signed measures $\mu_1,\ldots,\mu_n$ are nonpositive over E_{n+1}°, while μ_i is nonnegative over E_i°, $(i = 1,\ldots,n)$. The obvious interpretation of this phenomenon

is that E_i° is the region chosen by agent i, ($i = 1,\ldots,n$),
while E_{n+1}° is the residue ~~which is~~ left vacant. All the preceding theorems remain valid for this situation.

A

6.7. Joint Control and Agent Measure Spaces

We now consider a few real generalizations. In the above analysis each agent obtains exclusive control of a Space-Time region. But in the real world there are numerous examples of joint control: partnerships, joint ventures, committee management, corporate stockholders. ~~etc.~~ How ~~to~~ is one to represent this?

It is not immediately obvious how to "split" a set E among several agents in proportion to their share of control. However, recall that with any set $E \subseteq A$ is associated its indicator function $I_E : A \to \{0,1\}$, ~~namely~~ viz., $I_E(a) = 1$ if $a \in E$, $= 0$ if $a \in A\backslash E$. There is a 1-1 correspondence between the subsets of A and the set of all 0-1 functions on A by this association. Thus, instead of representing a real-estate allocation by $(E_1,\ldots,E_n)$ — (these forming a partition of A) — we could just as well have represented it by the corresponding n-tuple of indicators $(I_{E_1},\ldots,I_{E_n})$. For any $a \in A$, we have

~~Note that,~~

$$I_{E_1}(a) + \ldots + I_{E_n}(a) = 1. \qquad (6.7.1)\ \cancel{(1)}$$

Indeed, (1) is just another way of saying that $\{E_1,\ldots,E_n\}$ is a partition.

The advantage of using indicators is that it suggests the proper generalization to joint control. Namely, with agent i is associated a measurable function $h_i : A \to [0,1]$, taking values in the closed interval of real numbers between 0 and 1. The intended interpretation is: that $h_i(a)$ is the fraction of point a controlled by agent i. These n functions must satisfy

$$h_1(a) + \ldots + h_n(a) = 1, \qquad (6.7.2)\ \cancel{(2)}$$

for all $a \in A$, since the total of all fractional controls must add to 1.

Exclusive control is precisely the special case in which all functions h_i take on only the values 0 or 1. They are then all indicator functions, and (2) reduces to (1).

The entire structure of the real-estate market model generalizes in a corresponding way. Instead of preference orderings over pairs (E, x), as in (5.3) ~~(3) of section 5~~, the various agents have preferences over pairs (h, x). If μ° is the rental signed measure, the rental for the control pattern given by h will be

$$\int_A h \, d\mu^\circ. \qquad (6.7.3)\ \cancel{(3)}$$

The special assumption we made, that $U_i(E, x)$ is of the form $\mu_i(E) - x$, where μ_i is a bounded signed measure, now becomes

$$U_i(h, x) = \int_A h \, d\mu_i - x,$$

again for a bounded signed μ_i. Market equilibrium is given by rental distribution μ° and a measurable n-tuple $(h_1^\circ,\ldots,h_n^\circ)$ satisfying (2), such that no agent i prefers any control function to h_i°, when confronted with μ°.

g19-10 When the only allowable functions h are indicators, everything reduces to the original model with exclusive control.

There is no difficulty extending this model to a countably infinite, or even uncountable, number of participants. In the former case we get an infinite series on the left of (2), which must converge to 1 for all $a \in A$. In the latter case we use the concept of summation of an arbitrary collection of numbers (Section 2.2). ~~(page):~~ For each $a \in A$, all but a countable number of the values $h(a)$ equal 0, and the remainder form an infinite series, again converging to 1.

13 ~~15~~ A slightly different approach uses the concept of a measure space of agents, (B,Σ',ν).[13] Here B is the set of agents. It comes supplied with a ~~sigma~~ σ-field, Σ', on which is defined measure ν. Intuitively, $\nu(E)$ gives the "influence" of the set of agents E. To pin down this notion, we need a corresponding generalization of the concept of allocation in the real-estate market A. In the finite case, this was an n-tuple of functions $(h_1,\ldots,h_n)$ satisfying (2). This may also be written as a single function

$$h:\{1,\ldots,n\}\times A \to [0,1]. \qquad (6.7.4)\ \cancel{(4)}$$

ll left

For the set of traders B, an allocation will be a measurable function

$$h: B \times A \to [0,1], \qquad (6.7.5) \ \cancel{(5)}$$

with the rough intuitive meaning: h(b, a) is the fraction of land at $a \in A$ controlled "per unit influence" of agent $b \in B$. This rather vague notion, and the one above, are explicated formally by the requirement:

$$\int_B h(b,a)\,\nu(db) = 1, \qquad (6.7.6) \ \cancel{(6)}$$

for all $a \in A$. Thus (6) and (5) generalize (2) and (4), respectively, and indeed reduce to them when B consists of just n points ("agents"), Σ' = all subsets, and ν is ~~the enumeration~~ counting measure.

The generality of the "measure space of agents" approach may be illustrated by the case where ν is nonatomic (all singleton sets {b} being measurable). Here no single agent, or even any countable number of agents, has positive influence.

17 ~~16~~ As Aumann points out,[14] this is exemplified in the concept of perfect competition, where each agent has negligible influence. The obscurities that remain in the present formalism are matched by the obscurities present in that popular concept.

For each agent $b \in B$ there is a preference order over pairs $(h(b,\cdot), x)$. Here $h(b,\cdot)$ is a function with domain A, and gives the control pattern for agent b. The rental "per unit influence" for this control is given by (3) just as above. An equilibrium in the real-estate market consists of a rental distribution, μ^0 —

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which is a bounded signed measure over (A, Σ) – and a measurable control function h^o of the form (5), such that (6) is satisfied, and such that, for all agents b (except possibly for a set of ν-measure zero), $h^o(b,\cdot)$ is at least as preferred as any other control $h(b,\cdot): A \to [0,1]$ by agent b.

This entire approach, using functions h to represent joint control, applies just as well to joint ownership. Furthermore, there is nothing that restricts its use to the real-estate market. Consider, for example, the distribution of physical assets among economic agents at some time-instant t ("distribution" may refer either to ownership or to control). In this case, the set A is a subset of $R \times S$, rather than of $S \times T$ as in the real-estate market. For each agent i we again have a function $h_i: A \to [0,1]$, with the interpretation: $h_i(r,s)$ is the fraction of resource-type r at location s controlled (or owned, as the case may be) by agent i, at time t.

~~One final fpoint.~~ Suppose an agent's control pattern is given by $h: A \to [0,1]$. If there is some natural "quantity" measure on the space (A, Σ), ~~then~~ the control pattern may be expressed in terms of a measure, rather than a point function. For example, let $A \subseteq R \times S$, and h_i describe the control of physical assets by agent i at time t. Let μ_t be the cross-sectional measure at time t (so that μ_t is over $(R \times S, \Sigma_r \times \Sigma_s)$, and $\mu_t(E \times F)$ is the total mass of resources of types E in region F at time t), and let μ_t' be μ_t restricted to A.

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Then

$$\mu_{it} = \int_{\wedge} h_{i\wedge}\, d\mu'_{t},$$

an indefinite integral over A, expresses the control pattern of i at t as a measure: $\mu_{it}(G)$ is the total mass of resource-type-location pairs in G controlled by agent i at time t, for all measurable $G \subseteq A \subseteq (R \times S)$.

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Again, let α be "areal" measure on $(S \times T, \Sigma_s \times \Sigma_t)$. (If S is taken as 3-space, ~~then~~ α may be ~~four~~ 4-dimensional Lebesgue measure, say in units of "cubic-feet-days".)

Let α' be α restricted to E_o, the portion of Space-Time ~~which is~~ on the market. Then

break at = only

$$\alpha_i = \int_{\wedge} h_{i\wedge}\, d\alpha',$$

an indefinite integral over E_o, expresses i's real-estate control in measure form: $\alpha_i(G)$ is the amount of "cubic-feet-days" in region G controlled by agent i, for all measurable $G \subseteq E_o \subseteq (S \times T)$.

This is all very well and sometimes useful, but note that ~~it should be noted that,~~ in our entire discussion of the real-estate market, it was never ~~once~~ necessary to mention or use the concept of areal measure. All that was needed to define an equilibrium were the preference orderings over pairs (E, x), ~~—~~ or, more generally, (h, x). And to prove existence and other properties of equilibrium, what was needed was a specialization of the form of these preference orders. None of this involves areal measure. (Preferences among regions will of course be influenced by the areal capacities of these regions, among other factors. But this does not gainsay the fact that, once preferences are given, areal measure *per se* plays no rôle. It is a fifth wheel). This point is important in connection with Alonso's theory, where areal measure plays a key rôle.

6.8. Comparison with Alonso's Theory

One of the leading theories of the real-estate market, and deservedly so, is that of William Alonso. We ~~shall~~ now compare his theory with the one presented above. Our conclusion will be that, when certain kinks ~~in it~~ are straightened out, it becomes a special case of our own.

In his book of 1964 and preceding publications, Alonso develops his theory in the context of a featureless plain with a single point of attraction, which may be thought of as the central business district of a city.[15] But his underlying real-estate model does not really depend on this context, and, indeed, in a later article he briefly indicates a generalization.[16] We ~~shall~~ concentrate on the book, while stressing the general features of the theory.

Two specializations may be noted at once. First, it is a theory of Space rather than Space-Time. This creates no difficulties of comparison when viewed formally. All the "regions" in our model may be thought of as subsets of S if desired, rather than subsets of S × T.

Second, Alonso's is a <u>pure control</u> theory. Only the behavior of potential tenants is analyzed in detail, while landlords are assumed to auction off their land passively to the highest bidder. Our own theory has, in effect, made the same simplification, which ~~of course~~ loses a number of important real-world phenomena: e.g., discrimination, market frictions, ownership preferences, ~~etc.~~ Our "intermediate level" model, in which the preferences of agent i are summarized in a utility function, (5.3) ~~(3) of section 5~~, of the form

$$U_i(E, x) \tag{6.8.1}$$

~~(1)~~

(where E is the region controlled, and x is total <u>net</u> expenditure on real-estate), does allow for one aspect of owner-controller interaction: The formation of real-estate prices affects the real wealth of existing landords, and this "wealth effect" influences their preferences over regions (which in turn affects prices — we have a simultaneous equations situation). However, we proved no theorems at this level of generality, but passed on to the "constant marginal utility of money" formulation, (5.5), ~~(5) of section 5~~:

$$V_i(E) - x,$$

in which wealth effects disappear.

Alonso comes up with a preference order for agent i of the form

$$U_i(s,p) \qquad \text{(6.8.2)} \; \cancel{(2)}$$

(iii) where s is the point ~~at which~~ where he locates, and p is the rent per acre ~~that~~ he must pay. (~~Actually,~~ In the Thünen context of the model, U_i depends on s only through its distance from the point of attraction, so that (2) is a context-free generalization of what appears in the text. The same is true of all the other functions that involving location ~~that we write below.)~~ written below.) The indifference surfaces of (2) are called bid-price curves.

19.12 ¶ ~~The~~ The function (2) bears comparison with (1). It is~~, in fact,~~ a sort of "single point" version of (1), the single location s contrasting with the region E, and the rent density p with total expenditure x.

¶ Function (2) is derived by Alonso from an underlying preference ordering of agent i. This takes two forms, depending on whether i is a consumer or a firm. We ~~shall just~~ consider the consumer case. Agent i is then assumed to have a preference ordering represented by the utility function

$$U_i'(s,\ q,\ z), \qquad \text{(6.8.3)} \; \cancel{(3)}$$

where s again is location, q is acres of land controlled at location s, and z is all other goods consumed except land. Let π be the prices of the goods z, k(s) the transportation expenditure incurred by the agent as a function of location, and y_i the given income of agent i. The following budget condition must ~~now~~ be satisfied:

$$y_i = k(s) + \pi z + pq. \qquad (6.8.4)\ \cancel{(4)}$$

Now suppose s and p are given. The maximum of (3) over pairs (q, z) ~~which~~ that satisfy (4) will then depend on (s, p). This is the function (2).

Before continuing the analysis, ~~it should be~~ noted that our utility function (1) can be derived from an underlying preference ordering in a similar manner. We did not do so before ~~in order~~ to avoid distracting attention from the essential features of the real-estate market. To facilitate comparison with (3) and (4) ~~above~~, we postulate a single location s through which all commerce between the controlled region E and the rest of the world occurs. (This plays the same rôle as Alonso's "front door". See below.) We then postulate a utility function

$$U_i'(s, E, z), \qquad (6.8.5)\ \cancel{(5)}$$

where E is the region controlled by agent i, and z is all other goods, as above. (Actually, z should be written as a signed measure over $R \times S \times T$. We do not do so in order to keep things as similar as possible to (3) and (4)). Also, let y_i be the agent's wealth in non-real-estate assets.

Suppose ~~now~~ we are given E, the region agent i chooses to control, and x, his total net expenditure on real estate. The following budget condition is analogous to (4):

$$y_i = k(s) + \pi z + x. \qquad (6.8.6) \quad \cancel{(6)}$$

All terms in (6) are measured in discounted dollars. The maximum of (5) over all (s, z) satisfying (6) will then depend on (E, x). This is the utility function (1).

Let us now return to the Alonso utility function (3), and compare it with our (5). The region E in (5) can of course be any measurable subset of the universe set. Thus it can be the union of any number of disconnected pieces, each of these being of more or less arbitrary size and shape. By contrast, (3) refers to location "at" a single point s (or, in the Thünen context, "at" a single distance from the point of attraction). Thus, there is no way of representing the preferences of an agent contemplating multiple locations — say a family wanting both a town house and a country house, or wanting both a house and a business site. There is also no way of representing preferences regarding shape of lot, as opposed to size, which is represented by q.[17]

A second point concerning (3) is logically more serious, because it involves an actual inconsistency. If the acreage of land controlled is positive ($q > 0$), it must be spread over more than a single point (and even over more than a single distance from the point of attraction). Hence there is no clear meaning to the concept of being located "at" the single point s (or "at" a single distance from the point of attraction).

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Alonso is well aware of this problem of the "extended point". Before looking at his methods for dealing with it, we ~~let us~~ contemplate an alternative method. This involves postulating a measure space of agents, (B, Σ', ν), with ν non-atomic, in the manner of ~~à la~~ Aumann. We ~~One~~ might argue intuitively as follows: Since any one agent now has zero influence, he may be thought of as located literally "at" the single point s, even though his "acreage per unit influence", q, is positive.

Whatever one thinks ~~things~~ of this argument, it is not difficult to write ~~out,~~ formally, the equilibrium conditions ~~that~~ resulting from it. Let $p^o: S \to$ reals be the equilibrium rent-density function. Each agent $b \in B$ maximizes (3) over s, q, z, subject to condition (4) with $p(s) = p^o(s)$. (Subscript i is replaced by b in (3) and (4).) Let $s(b)$, $q(b)$ be the locational and acreage choices of agent b. Then we must have, for any region E,

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[52]
$$\int_{\{b \mid s(b) \in E\}} q \, d\nu = \alpha(E). \qquad (6.8.7)\ \cancel{(7)}$$

which ~~(7)~~ states that $\alpha(E)$, the total area of E, must equal the total demand for land by those agents who locate in E.

Alonso does not take this tack. Indeed, the Aumann approach requires an uncountable number of agents, while Alonso always works with a finite number. Instead, he makes an ad hoc assumption ~~which~~ depending essentially on the Thünen context; namely, that (s, q) refers to a ring-shaped region of area q,

(w)

concentric with the point of attraction, whose inner radius is the distance of s from the point of attraction. (This assumption is dropped in his Appendix B).

The "extended point" problem causes one other bit of trouble. ~~Namely,~~ since there is no single location s "at" which the agent chooses, there will in general not be any single rent density p(s) either. This means that the term pq in (4) must be replaced by an integral, giving total expenditure for control of the ring (s, q) chosen by the agent. (This is done in his Appendix A). It also means that the derived utility function $U_i(s, p)$ — which is (2) ~~above~~ — with its "bid-price" level surfaces, has no clear meaning.

~~Finally,~~ in his Appendix B ~~of his book,~~ Alonso postulates a more general utility function than (3) ~~in order~~ to take account of shape and avoid making the *ad hoc* assumption just mentioned. The agent i chooses a point-location s, called his "front door", and a region E to control. His preferences are expressed by the utility function

12 pt brackets

$$U_i'\left[s, \int_E f(d'(s,y))\,\alpha(dy), z\right]. \qquad \text{(6.8.8)}\ \cancel{(8)}$$

The q-argument in (3) has been replaced by an integral, the other two arguments, s and z, remaining the same. α is, as usual, areal measure. Now q can be expressed in terms of region E: $q = \alpha(E) = \int_E 1\, d\alpha$ — so that (8) can be thought of as a generalization of (3) in which 1 has been replaced by a non-constant integrand. In this integrand, d' is the (Euclidean) metric on the plane, and f is some positive, strictly decreasing, function. In effect, (8) makes the value of a point y inversely related to its distance from the agent's "front door".

With this change, Alonso's model becomes a special case of our own, for (8) is a specialization of the utility function (5). The budget condition in both cases is given by (6). ~~As described above,~~ we can thus obtain the derived utility function (1): $U_i(E, x)$, etc.

We now give an informal, non-rigorous argument to indicate how (8) gives information concerning the shape of the region E. Let rent density function $p^o: S \to$ positive reals be given. Suppose s, z, and x are chosen in advance. Subject to these values and to the budget constraint (6), we are to choose E to maximize (8). Any conditions derived from this problem will be necessary for optimality in the original problem. Assuming (8) to be increasing in its middle argument,

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one easily verifies that this special problem simplifies to:

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Maximize

do not break

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$$\int_E f(d'(s,y))\,\alpha(dy)$$

over E, subject to

$$\int_E p^o\, d\alpha = x.$$

This can be turned into an allocation-of-effort problem ~~such~~ as ~~we considered~~ in chapter 5. Omitting details, the optimal set E° has the following form:

$$E^{\circ} = \{y \mid f(d'(s,y))/p^{\circ}(y) \geq c\}, \qquad (6.8.9)\ \cancel{(9)}$$

for some constant c. Along the borderline of E°, the inequality of (9) becomes an equality.

Alonso derives this condition — that the ratio of $f(d'(s,y))$ to $p^{\circ}(y)$ is constant along the borderline. But at this point his analysis falters. Assuming rent density p° to be a decreasing function of distance from the point of attraction, he concludes that the optimal region is *egg-shaped*. (Recall that we are on the Euclidean plane.) But this is impossible in market equilibrium, since the plane — or any circular disc on it — cannot be partitioned into egg-shaped regions. The correct conclusion is that utility functions of the form (8) *preclude* the possibility that rent density has the property just mentioned. On the contrary, the real-estate market equilibrium (if it exists) must be such that, with the given p°, the agents choose regions ~~which~~ that partition the market among themselves.

In conclusion, in its most developed form, Alonso's model becomes a special case of our own. Earlier versions involve either inconsistencies or *ad hoc* assumptions.

The following program remains to be carried out. For plausible utility functions $U_i(E, x)$, find the conditions under which a real-estate market equilibrium exists, describe its form, determine the equilibrium rental measure μ^o, and find any interesting extremal or other properties that the equilibrium possesses.

Among the plausible utility functions are those derived from an underlying utility of the form (8), with budget constraint (6). Since Alonso's exposition is flawed, it is not clear under what conditions an equilibrium even exists. More general functions than (8) should also be considered. For one thing, it seems unlikely that an agent with preferences represented by (8) would ever choose a region of multiple scattered locations, a very common real-world phenomenon. Rather, he would choose a region tightly clustered around his "front door". To generalize, ~~one~~ we need~~s~~ the possibility of numerous "front doors", or perhaps an entirely different form of preference ordering.

~~The only case in which the~~ This program has been carried out in two cases. The first is where $U_i(E, x) = \mu_i(E) - x$, for some bounded signed measure μ_i (agents $i = 1,\ldots,n$), which yields the theory of section 6.6 (and which~~, in fact,~~ we generalize in ~~the following~~ section 6.9). ~~The only trouble is that this form of the utility function is not very realistic, as our previous critique indicated.~~ The second case is the Thünen equilibrium of section 8.7. ~~below~~

μ

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(A)

6.9. Pseudomeasure Treatment of the Real-Estate Market

~~Let us~~ We return to the real-estate equilibrium model of section 6.2.1 and ~~f~~For convenience ~~we~~ repeat it here. We are given a measurable space, (A,Σ), and n bounded signed measures, $\mu_1,\ldots,\mu_n$, one for each agent in the market.

An equilibrium consists of a rental distribution, μ°, which is a bounded signed measure on (A,Σ), and an n-tuple of measurable sets, $(E_1^\circ,\ldots,E_n^\circ)$, ~~which~~ that partition A, and are such that E_i° maximizes

$$\mu_i(E) - \mu^\circ(E) \qquad (6.9.1)$$

over all $E \in \Sigma$, for each $i = 1,\ldots,n$. The interpretation, of course, is that E_i° is the region chosen by agent i, while (1) is the utility he attaches to the control of region E, $\mu^\circ(E)$ being the net expenditure he incurs for this control.

We want to generalize this set-up in two directions. First, to consider the case of a (countably) *infinite* number of agents. Second, to consider what happens if μ°, or some or all of the μ_i, are *infinite* (~~sigma~~ σ-finite) signed measures.

Before launching into details, ~~let us~~ we briefly discuss the question of whether such generalizations have any possible applications. (~~W~~we use the term "applications" as usual in a rather liberal sense). Consider models with unbounded Space or Time horizons. On the "endless plain" of location theory, for example, where the same patterns are repeated indefinitely (as in the Löschian system), we may reasonably presume that the

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use bastard italic level for le static pres number of agents, and total rentals, will be infinite. With an unbounded Time horizon we may have an infinite sequence of generations. Whether rentals become infinite with unbounded time is more dubious (recall that everything is measured in discounted dollars, so that the present value of total rentals may well be finite even for unbounded Time).

It is a little harder to find a rationale for the μ_i in the preferences of the individual agents to be infinite. Nonetheless, such a preference order might be reasonable for organizations such as corporations or governments ~~which~~ are potentially immortal, or ~~which~~ can extend their control indefinitely over infinite Space. Furthermore, there are other interpretations. If the i's are interpreted as activities or land uses rather than as agents, ~~then~~ the real-estate equilibrium may be thought of as the result of a global competition among alternative uses for the allocation of Space-Time. In this case, under the appropriate world-system, the μ_i may well be infinite.

~~Let us~~ We now proceed to the generalizations. An immediate difficulty arises. In (1) the meaningless expression $\infty - \infty$ may arise for certain values of E. Also there may be several values of E yielding $+\infty$ (or $-\infty$) ~~in (1)~~. Are these to be considered indifferent, or is it possible to discriminate among them?

The reader familiar with chapter 3 will notice we have a situation tailor-made for the application of pseudomeasures. Namely, we interpret the difference in (1) in the sense of pseudomeasures, and interpret "maximization" of (1) as referring to one or another of the orderings on the space of pseudomeasures discussed in chapter 3.

When this is done, ~~it turns out that~~ the entire theory of section 6 generalizes directly from bounded signed measures to pseudomeasures. ~~Also~~ (with the exception of one theorem) it also generalizes from a finite to a countable number of agents. Things generalize not only theorem by theorem, but even proof by proof, so that ~~one~~ we might say that the natural realm in which

18 ~~70~~ the theory of Section 6.6 is valid is the realm of pseudomeasures.[18]

In what follows we ~~shall~~ make still a further generalization and ~~We shall~~ take not merely the differences $\mu_i - \mu^\circ$, but also μ_i and μ° themselves, to be pseudomeasures. This may have direct application, in case there are "infinitely dispreferred" as well as "infinitely preferred" regions. However, our main object in doing this is that it facilitates the following development to plunge completely into the realm of pseudomeasures: Proofs are smoother and theorems are simpler to state (as well as more general) ~~of course)~~.

Before ~~getting started~~ proceeding, we need a convention and one or two definitions. The convention arises from our dealing with a countable, rather than a finite, number of agents. Formerly, we had an n-tuple of signed measures $(\mu_1,\ldots,\mu_n)$, and a corresponding n-tuple of sets $(E_1,\ldots,E_n)$. Now we have a sequence of pseudo-measures $(\psi_1, \psi_2,\ldots)$, and a corresponding sequence of sets

$(E_1, E_2, \ldots)$. These sequences are countable, possibly finite ~~— that is, either finite or infinite~~. We now insist that these two sequences be of the same length: either both infinite, or both finite of length n (so that there is in any case a 1-1 correspondence between ψ_i and E_i). Both these cases will be encompassed by the single notation "$(\psi_1, \psi_2, \ldots)$, $(E_1, E_2, \ldots)$". These sequences are of finite or infinite, but in any case equal, length.

We give ~~Now for the definitions~~. We refer to the triple (A, Σ, ψ) as a pseudomeasure space iff (A, Σ) is a measurable space, and ψ a pseudomeasure on this space. Sometimes we ~~shall~~ use the notation (A, Σ, μ, ν) for the same thing, where (μ, ν) is any representation of the pseudomeasure ψ as a pair of ~~sigma~~ σ-finite measures.

Definition: Let (A, Σ, ψ) be a pseudomeasure space, and let E_1 be a measurable subset of A. The restriction of ψ to E_1 is the pseudomeasure space (E_1, Σ_1, ψ_1), where Σ_1 is Σ restricted to subsets of E_1, and $\psi_1 = (\mu_1, \nu_1)$; here μ_1 and ν_1 are the restrictions of μ and ν to E_1, respectively, where (μ, ν) is any representative of ψ.

That is, given ψ, choose any one of its forms (μ, ν); restrict the two measures to E_1, getting μ_1 and ν_1; and then ψ_1 is the pseudomeasure to which the pair (μ_1, ν_1) belongs.

It must be shown that this is a bona fide definition; ~~that is~~ i.e., the resulting pseudomeasure ψ_1 must be the same no matter what pair representing ψ is chosen. This is easily verified. The basic equivalence theorem for pseudomeasures states that $(\mu,\nu) = (\mu',\nu')$ (~~that is~~ i.e., they represent the same pseudomeasure) iff

$$\mu + \nu' = \nu + \mu'. \qquad (6.9.2)$$

Now suppose $\psi = (\mu,\nu) = (\mu',\nu')$, so that (2) holds. Restricting all measures to E_1, we ~~obviously~~ have $\mu_1 + \nu_1' = \nu_1 + \mu_1'$, so that $\psi_1 = (\mu_1,\nu_1) = (\mu_1',\nu_1')$. Thus the definition is sound.

Sometimes we ~~shall~~ refer to ψ_1 itself, instead of (E_1,Σ_1,ψ_1), as the restriction of ψ to E_1.

Next, we need the concept of a direct sum of pseudomeasures. ~~First~~ recall what this concept means for measures. Let (E_i, Σ_i, μ_i), $i = 1, 2,\ldots,$ be a (finite or infinite) sequence of measure spaces, the universe sets $E_1, E_2,\ldots$ being mutually disjoint. The direct sum of these spaces, written

$$(E_1, \Sigma_1, \mu_1) \oplus (E_2, \Sigma_2, \mu_2) \oplus \ldots \qquad (6.9.3)$$

or $\oplus_i (E_i, \Sigma_i, \mu_i)$, is the measure space whose universe set $\oplus_i E_i = E_1 \cup E_2 \cup \ldots$, whose σ-~~sigma-~~field $\oplus_i\Sigma_i$ consists of all sets of the form $F_1 \cup F_2 \cup \ldots$, where $F_i \in \Sigma_i$, $i = 1, 2,\ldots,$ and whose measure is given by

$$(\oplus_i \mu_i)(F_1 \cup F_2 \cup \ldots) = \mu_1(F_1) + \mu_2(F_2) + \ldots . \qquad \cancel{(4)}\ (6.9.4)$$

Definition: Let (E_i, Σ_i, ψ_i), $i = 1, 2, \ldots$, be a (finite or infinite) sequence of pseudomeasure spaces, the universe sets $E_1, E_2, \ldots$ being mutually disjoint. The direct sum of these is the pseudomeasure space $(\oplus_i E_i, \oplus_i \Sigma_i, \oplus_i \psi_i)$, where $\oplus_i E_i$ and $\oplus_i \Sigma_i$ are defined just as above, while $\oplus_i \psi_i = (\oplus_i \mu_i, \oplus_i \nu_i)$, (μ_i, ν_i) being any representative of ψ_i, $i = 1, 2, \ldots$.

That is, for each ψ_i, choose any one of its forms (μ_i, ν_i), take the direct sum of the sequence of measures $\mu_1, \mu_2, \ldots$, and do the same for $\nu_1, \nu_2, \ldots$; this yields a pair of measures, and $\oplus_i \psi_i$ is the pseudomeasure to which this pair belongs.

Again, it must be shown that this is a bona fide definition. We note, first of all, that the direct sum of ~~sigma~~ σ-finite measures is a ~~sigma~~ σ-finite measure. To see this, take the direct sum in (3), with each μ_i ~~sigma~~ σ-finite. Each E_i then has a countable measurable partition $\{E_{i1}, E_{i2}, \ldots\}$ such that $\mu_i(E_{ij})$ is finite, all $i, j = 1, 2, \ldots$. The collection of all the sets E_{ij} is then a countable measurable partition of $\oplus_i E_i$; furthermore, by (4), μ_i coincides with $\oplus_i \mu_i$ on E_{ij}. Hence the latter measure is ~~sigma~~ σ-finite.

This proves ~~that~~ the pair $(\oplus_i \mu_i, \oplus_i \nu_i)$ does indeed represent a pseudomeasure. We must now show that the resulting pseudomeasure is the same no matter what pair representing each component ψ_i is chosen. To prove this, for each i let (μ_i', ν_i') be another pair representing ψ_i. By the equivalence theorem,

$$\mu_i + \nu_i' = \nu_i + \mu_i' \qquad (6.9.5)\ \cancel{(5)}$$

for each $i = 1, 2, \ldots$. Taking direct sums, we obtain

$$(\oplus_i \mu_i) + (\oplus_i \nu_i') = \oplus_i (\mu_i + \nu_i') \qquad (6.9.6)\ \cancel{(6)}$$
$$= \oplus_i (\nu_i + \mu_i') = (\oplus_i \nu_i) + (\oplus_i \mu_i').$$

The middle equality in (6) comes from (5); the first and last equalities are easy consequences of (4). Hence, again using the equivalence theorem, we obtain

$$(\oplus_i \mu_i', \ \oplus_i \nu_i) = (\oplus_i \mu_i', \ \oplus \nu_i').$$

Hence the direct sum $\oplus_i \psi_i$ is, indeed, well-defined.

18 Now, let a fixed measurable space (A, Σ) be given, and let $(\psi_1, \psi_2, \ldots)$ be a sequence of pseudomeasures on (A, Σ). For any corresponding sequence of measurable sets $(E_1, E_2, \ldots)$ ~~which~~ that partition A, we define a new pseudomeasure on (A, Σ) as follows. First, restrict each ψ_i to its corresponding set E_i. (Recall our convention concerning equality of length of these sequences). Second, take the direct sum of these restrictions. The result is a pseudomeasure ~~which~~ that is, intuitively, obtained by patching together pieces of the original ψ_i's. We ~~shall~~

denote this pseudomeasure by ψ_E, so that

$$\psi_E = (\psi_1|E_1) \oplus (\psi_2|E_2) \oplus \ldots, \qquad (6.9.7)\ \cancel{(7)}$$

$\psi_i|E_i$ being the restriction of ψ_i to E_i.

This construction bears a strong resemblance to (6.16) ~~the expression (16) of section 6.~~ Recall that (6.16) ~~(16)~~, which is $\mu_1(E_1) + \ldots + \mu_n(E_n)$, was the social valuation of the allocation $(E_1,\ldots,E_n)$, each agent placing his own evaluation (in discounted dollars) on the region he controls. Formula (7) is the natural generalization of (6.16) ~~(16)~~, with everything now being in terms of pseudomeasures rather than numbers.

The last theorem of section 6. stated that the n-tuple $(E_1^\circ,\ldots,E_n^\circ)$ partitioning A maximized social valuation iff it was an extended Hahn decomposition for the n-tuple of signed measures $(\mu_1,\ldots,\mu_n)$. Is there a corresponding generalization involving (7)?

The answer is yes, ~~provided~~ if things are defined in the right way. First, we must generalize the decomposition concept to pseudomeasures. Second, we must specify in what sense(s) the term "maximization" is to be understood: What ordering is be~~ing~~ in referred to — narrow ordering of pseudomeasures? standard ordering? etc., and is the "maximizer" greatest, or merely unsurpassed? ~~In fact,~~ We get a richer, as well as more general, theorem by distinguishing these alternative meanings.

Definition: Let (A,Σ) be a measurable space, and $(\psi_1, \psi_2, \ldots)$ a sequence of pseudomeasures on (A,Σ). The corresponding sequence of measurable sets $(E_1^\circ, E_2^\circ, \ldots)$ is an extended Hahn decomposition for $(\psi_1, \psi_2, \ldots)$ iff $\{E_1^\circ, E_2^\circ, \ldots\}$ partitions A, and, for all $i, j = 1, 2, \ldots,$

$$(\psi_i - \psi_j)^-(E_i^\circ) = 0. \tag{8}$$

(6.9.8)

Here ψ^- denotes the lower variation of ψ, just as ψ^+ denotes the upper variation. (8) states that ψ_i is at least as large as ψ_j (in the sense of narrow ordering of pseudomeasures) when both are restricted to E_i°. In the case when ψ_i and ψ_j are both bounded signed measures, (8) reduces to ~~(2) of section 6~~ (6.2); hence this is indeed a generalization of the same concept defined in section 6.

Now let $(\psi_1, \psi_2, \ldots)$ be a sequence of pseudomeasures on measurable space (A,Σ). We consider all possible corresponding sequences of measurable sets $(E_1, E_2, \ldots)$ such that $\{E_1, E_2, \ldots\}$ partitions A. With each such sequence is associated a pseudomeasure ψ_E by the rule (7), giving us a set of pseudomeasures, Ψ_E. Let $(E_1^\circ, E_2^\circ, \ldots)$ be one particular such sequence, with its associated pseudomeasure $\psi_{E^\circ} \in \Psi_E$.

l.c. ψ cap Ψ

Theorem: Each of the following five statements implies the other four:

(i) $(E_1^\circ, E_2^\circ, \ldots)$ is an extended Hahn decomposition for $(\psi_1, \psi_2, \ldots)$;

(ii) ψ_{E° is <u>greatest</u> in the set Ψ_E, in the sense of narrow ordering of pseudomeasures;

(iii) ψ_{E° is <u>greatest</u> in Ψ_E, in the sense of standard ordering;

(iv) ψ_{E° is <u>unsurpassed</u> in Ψ_E, in the sense of standard ordering;

(v) ψ_{E° is <u>unsurpassed</u> in Ψ_E, in the sense of narrow ordering.

<u>Proof</u>: (ii) implies (iii) implies (iv) implies (v): These follow at once from the fact that standard ordering extends narrow ordering, and from the definitions of "greatest" and "unsurpassed".

To complete the proof, we show that (i) implies (ii), and (v) implies (i).

Let (i) be true, and let $(E_1, E_2, \ldots)$ be another feasible sequence. For any $i, j = 1, 2, \ldots$, it follows from (7) that ψ_{E° restricted to E_i° is the same as ψ_i restricted to E_i°, and ψ_E restricted to E_j is the same as ψ_j restricted to E_j. Hence

$$(\psi_{E^\circ} - \psi_E)^-(E_i^\circ \cap E_j) = (\psi_i - \psi_j)^-(E_i^\circ \cap E_j) = 0. \qquad (6.9.9)$$

(The last equality in (9) follows from (8)).

Summing (9) over all pairs (i, j), where i, j range independently over $1, 2, \ldots$, we obtain

$$(\psi_{E^\circ} - \psi_E)^-(A) = 0,$$

~~which is~~ so that, by definition, $\psi_{E^\circ} \geq \psi_E$ (narrow order). Since ψ_E was an arbitrary member of Ψ_E, ψ_{E° is greatest under narrow ordering.

This proves that (i) implies (ii).

\# 18 The last part is a little more difficult. Assume that (i) is false, so that ~~there~~ exist indices m, n for which

$$(\psi_m - \psi_n)^-(E_m^\circ) > 0. \qquad \cancel{(10)}\ (6.9.10)$$

We ~~shall~~ now construct a feasible sequence $(E_1, E_2, \ldots)$ whose associated pseudomeasure ψ_E surpasses ψ_{E° (narrow order), proving (v) false.

Let (P,N) be a Hahn decomposition for the pseudomeasure $\psi_m - \psi_n$. Define $(E_1, E_2, \ldots)$ by:

$$E_m = E_m^\circ \cap P, \quad E_n = E_n^\circ \cup (E_m^\circ \cap N), \quad E_i = E_i^\circ \text{ for all } i = 1, 2, \ldots \text{ other than } i = m \text{ or } i = n. \qquad \cancel{(11)}\ (6.9.11)$$

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Note that $m \neq n$, from (10). Hence (11) is well-defined and the sequence $(E_1, E_2, \ldots)$ so defined is measurable and partitions A.

Pseudomeasure ψ_E derived from this sequence coincides with ψ_n when both are restricted to E_n. Also, ψ_{E° coincides with ψ_m when both are restricted to E_m°. Now $E_m^\circ \cap N$ is a subset of both E_n and E_m°. Hence

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$$(\psi_{E^\circ} - \psi_E)^-(E_m^\circ \cap N) = (\psi_m - \psi_n)^-(E_m^\circ \cap N) = (\psi_m - \psi_n)^-(E_m^\circ) > 0. \qquad \cancel{(12)}\ (6.9.12)$$

fl left

The first equality in (12) arises from substitution; the second from the fact that (P,N) is a Hahn decomposition of $\psi_m - \psi_n$; the inequality is (10).

Now (12) implies that $\psi_{E^\circ} \not\geq \psi_E$ (narrow order). It remains to show that $\psi_E \geq \psi_{E^\circ}$, which will prove that ψ_{E° is surpassed. We do this by proving that

$$(\psi_E - \psi_{E^\circ})^-(E_i) = 0 \qquad \text{(6.9.13)} \quad \cancel{(13)}$$

for all $i = 1, 2, \ldots$. Equality (13) is in fact immediate for all i other than $i = n$, since, when restricted to E_i, both ψ_E and ψ_{E° coincide with ψ_i, hence with each other.

This leaves E_n; we consider its two pieces, E_n° and $(E_m^\circ \cap N)$, separately. On E_n°, ψ_E and ψ_{E° again both coincide with ψ_n. On $(E_m^\circ \cap N)$, ψ_{E° coincides with ψ_m, and ψ_E with ψ_n. Hence

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$$(\psi_E - \psi_{E^\circ})^-(E_m^\circ \cap N) = (\psi_n - \psi_m)^-(E_m^\circ \cap N) \leq (\psi_n - \psi_m)^-(N) = 0.$$

Hence (13) is true for all i. Adding over i, we obtain

$$(\psi_E - \psi_{E^\circ})^-(A) = 0,$$

which is: $\psi_E \geq \psi_{E^\circ}$. Combined with $\psi_{E^\circ} \not\geq \psi_E$, we find that ψ_{E° is surpassed, so (v) is false. Hence (v) implies (i), ~~and the proof is complete.~~ ### □

The corresponding theorem in section 6. asserts the equivalence of $(E_1^\circ, E_2^\circ, \ldots)$ being an extended Hahn decomposition and maximizing $\mu_1(E_1) + \ldots + \mu_n(E_n)$. These statements are specializations of (i), and (iii) or (iv), respectively: The maximization refers only to "standard ordering" in the realm of bounded signed measures; i.e. ~~— that is~~, based on the value assigned to the universe set A. By adding the specializations of statements (ii) and (v), we get a stronger theorem in the realm of bounded signed measures. Thus we find that an extended Hahn decomposition maximizes social valuation not only on A, but on every measurable set simultaneously; this follows from (i)'s implying (ii).

To the five logically equivalent statements just mentioned two more can be added; ~~namely,~~ any of the five statements implies, and is implied by, the condition that "ψ_{E° is greatest in the set Ψ_E under any extended ordering of pseudomeasures". This is the first statement; the second is obtained by replacing "greatest" by "unsurpassed". These may be inserted in the chain of implications between statements (iii) and (iv); this follows from the fact that any extended ordering is an extension of standard ordering.

The economic interpretation of this theorem is the same as in section 6. It expresses the extended Hahn decomposition property as an extremal property. Since, as will next be shown, a sequence $(E_1, E_2, \ldots)$ is such a decomposition for $(\psi_1, \psi_2, \ldots)$

iff it is a market equilibrium, it also will have demonstrated an equivalence between market equilibrium and the maximization of social valuation, just as in section 6.

Let us ~~now~~ return to the real-estate market. We now have a (finite or infinite) sequence of agents. A market equilibrium will consist of a corresponding sequence of measurable sets $(E_1^\circ, E_2^\circ, \ldots)$ and a rental pseudomeasure ψ° on (A, Σ). Here E_i° will ~~of course~~ be ~~that~~ the region falling to ~~the~~ control of agent i, and these sets must partition A. Agent i will prefer region E_i° at least as well as any other region, given ψ°.

But here a slight problem arises. How should the preferences of agent i over regions be represented? In the bounded signed measure case, agent i had the utility function

$$U_i(E) = \mu_i(E) - \mu^\circ(E) \qquad (6.9.14)\ \cancel{(14)}$$

In the general case one naturally expects utility to be pseudomeasure-valued (and to reduce to (14) when all pseudomeasures are in fact bounded signed measures). One's first impulse is to assign to set E a pseudomeasure restricted to E. But this will not do: pseudomeasures are comparable only if they are defined on the same measurable space. Thus if $U_i(E)$ were a pseudomeasure with universe set E, no two regions would be comparable.

20.6 This difficulty is easily resolved: ~~Namely,~~ with region E we associate the pseudomeasure

$$[(\psi_i - \psi^\circ)|E] \oplus [0|(A\backslash E)] \qquad (6.9.15)\ \cancel{(15)}$$

fl left

This is the direct sum of $(\underline{\psi_i} - \psi^o)$ restricted to $\underline{E}$, and the <u>zero</u> pseudomeasure restricted to $\underline{A}\backslash\underline{E}$. All pseudomeasures (15) are over the space $(\underline{A},\Sigma)$. Pseudomeasure $\underline{\psi_i}$ is, ~~of course,~~ the generalization of $\underline{\mu_i}$, and may be thought of intuitively as giving the value to agent $\underline{i}$ of the various regions, gross of any rental outlay. (More exactly, the pseudomeasure $[\underline{\psi_i}|\underline{E}]\oplus[0|(\underline{A}\backslash\underline{E})]$ gives $\underline{i}$'s gross evaluation of region $\underline{E}$).

One easily verifies that, if $\underline{\psi_i}$ and ψ^o are both bounded signed measures, ~~then~~ (15) under standard ordering in effect reduces to (14), with $\underline{\mu_i} = \underline{\psi_i}^+ - \underline{\psi_i}^-$, $\underline{\mu^o} = (\psi^o)^+ - (\psi^o)^-$. ~~(That is,~~ i.e., both (15) and (14) determine the same preference ordering over regions in this case). Thus (15) seems to be the natural generalization of (14).

Consider the result of adding $\underline{\psi^o}$ to (15). It is

$$[\underline{\psi_i}|E] \oplus [\underline{\psi^o}|(\underline{A}\backslash E)]. \qquad \text{(6.9.16)} \; \cancel{(16)}$$

The verification of (16) rests on two observations. First, that

$$\psi = [\psi|\underline{E}] \oplus [\psi|(\underline{A}\backslash\underline{E})] \qquad \text{(6.9.17)} \; \cancel{(17)}$$

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is an identity for any pseudomeasure, in particular for $\underline{\psi^o}$. Second, that

$$[\underline{\psi_i}|E] \oplus [\underline{\psi^o}|(A\backslash E)] = \Big([(\underline{\psi_i} - \underline{\psi^o})|E] \oplus [0|(A\backslash E)]\Big) + \Big([\underline{\psi^o}|E] \oplus [\underline{\psi^o}|(A\backslash E)]\Big), \qquad \text{(6.9.18)} \; \cancel{(18)}$$

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which again is a special case of a theorem concerning sums of direct sums. The verification of (17) and (18) is left as an exercise. The corresponding transformation of (14) is

$$\mu_i(E) + \mu^\circ(A\backslash E). \qquad (6.9.19)\ \cancel{(19)}$$

~~Now,~~ since (15) and (16) differ only by a constant, they determine the same ordering over regions E, whether we use narrow order or standard order (or any other partial order on the vector space of pseudomeasures ~~which~~ that is determined by a convex cone). This follows from the fact that, for any three pseudomeasures ψ°, ψ^1, ψ^2 over (A,Σ),

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$$\psi^1 \succcurlyeq \psi^2 \text{ iff } \psi^1 + \psi^\circ \succcurlyeq \psi^2 + \psi^\circ,$$

where "$\succcurlyeq$" stands for any such partial ordering.

Hence the utility of region E for agent i could just as well be given by (16) as by (15). (Similarly, in the special case of bounded signed measures, $U_i(E)$ given by (19) yields the same preference ordering over regions as $U_i(E)$ given by (14)).

Now let $(E_1^\circ, E_2^\circ,\ldots)$, ~~together~~ with ψ°, be a real-estate market equilibrium. This means that, for each i, E_i° maximizes (15), or, equivalently, (16). But what does "maximize" mean? Does it refer to narrow or standard ordering, and does E_i° maximize in the sense of being greatest, or merely unsurpassed?

Our preceding theorem furnishes a complete and satisfying answer to these questions. ~~Namely,~~ For the utility functions (15) or (16), all ~~of~~ these senses of the term "maximize" are equivalent: E_i^o maximizes in one of these senses iff it maximizes in any other sense. Furthermore, by the same theorem, a necessary and sufficient condition that E_i^o maximize in any (hence all) of these senses, is that the pair $(E_i^o, A\backslash E_i^o)$ be an extended Hahn decomposition for the pair of pseudomeasures (ψ_i, ψ^o).

The preceding theorem refers to sequences of pseudomeasures, $(\psi_1, \psi_2, \ldots)$, and corresponding sequences of measurable sets, $(E_1, E_2, \ldots)$ ~~which~~ that partition A. The statements above are nothing but the special case in which these sequences are merely pairs. The statement that $(E_i^o, A\backslash E_i^o)$ is a decomposition for the pair (ψ_i, ψ^o) comes from the utility function (16). If instead we use (15), we find that, equivalently, it is an extended Hahn decomposition for the pair $(\psi_i - \psi^o, 0)$. Furthermore, it follows at once from the definitions that these statements are true iff $(E_i^o, A\backslash E_i^o)$ is a Hahn decomposition for the pseudomeasure $\psi_i - \psi^o$. Thus we have seven or eight logically equivalent conditions on the set E_i^o.

With these preliminary comments out of the way, we now state a result ~~which~~ that directly generalizes the first theorem of section 6.

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Theorem: Let $(\psi_1, \psi_2, \ldots)$ be a sequence of pseudomeasures on space (A, Σ), and let $(E_1^o, E_2^o, \ldots)$ be a corresponding sequence of measurable sets ~~which~~ that partition A. Then pseudomeasure ψ^o

satisfies one of the following two conditions iff it satisfies the other:

(i) For each $i = 1, 2, \ldots$, $(E_i^\circ, A \setminus E_i^\circ)$ is an extended Hahn decomposition for the pair (ψ_i, ψ°);

(ii) for all $i, j = 1, 2, \ldots$ (with $i \neq j$) we have

$$(\psi_i | E_i^\circ) \geq (\psi^\circ | E_i^\circ) \geq (\psi_j | E_i^\circ). \qquad \text{(6.9.20)} \ \cancel{(20)}$$

(~~That is~~ i.e., when all are restricted to E_i°, the three pseudo-measures are narrowly ordered as indicated).

Furthermore, ~~there~~ exists a ψ° satisfying one (hence both) of these conditions iff $(E_1^\circ, E_2^\circ, \ldots)$ is an extended Hahn decomposition for $(\psi_1, \psi_2, \ldots)$.

Proof: Condition (ii) may be rewritten as:

$$(\psi_i - \psi^\circ)^-(E_i^\circ) = 0, \quad (\psi^\circ - \psi_j)^-(E_i^\circ) = 0, \qquad \text{(6.9.21)} \ \cancel{(21)}$$

while condition (i) is

$$(\psi_i - \psi^\circ)^-(E_i^\circ) = 0, \quad (\psi^\circ - \psi_i)^-(E_j^\circ) = 0, \qquad \text{(6.9.22)} \ \cancel{(22)}$$

both holding for all $i, j, i \neq j$. The left conditions in (21) and (22) are identical. The right conditions are also identical except for the interchange of indices i and j. This proves conditions (i) and (ii) imply each other.

Next, suppose ψ° exists satisfying these conditions. From (20) we obtain

$$(\psi_i - \psi_j)^-(E_i^\circ) = 0 \qquad (6.9.23)\ \cancel{(23)}$$

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for all i, j (with $i \neq j$). But (23) is exactly the condition (8) that $(E_1^\circ, E_2^\circ, \ldots)$ be an extended Hahn decomposition for $(\psi_1, \psi_2, \ldots)$.

Finally, suppose that $(E_1^\circ, E_2^\circ, \ldots)$ is an extended Hahn decomposition for $(\psi_1, \psi_2, \ldots)$. Let ψ° be the pseudomeasure

$$(\psi_1 | E_1^\circ) \oplus (\psi_2 | E_2^\circ) \oplus \ldots . \qquad (6.9.24)\ \cancel{(24)}$$

We then obtain

$$(\psi^\circ | E_i^\circ) = (\psi_i | E_i^\circ) \geq (\psi_j | E_i^\circ), \qquad (6.9.25)\ \cancel{(25)}$$

for all i, $j = 1, 2, \ldots$. The equality in (25) comes from (24); the inequality is the same as the decomposition condition (23). Since (25) implies (20), the proof is complete. ~~///~~ □

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According to the discussion preceding this theorem, condition (i) holds iff E_i° maximizes the utility of agent i, (15) or (16), for all i, i.e. ~~that is,~~ iff $(E_1^\circ, E_2^\circ, \ldots)$, 1 & 2 combined with ψ°, is a real-estate equilibrium. Hence this theorem states that the condition of being a real-estate equilibrium is logically equivalent to being an extended Hahn decomposition, for a given sequence $(\psi_1, \psi_2, \ldots)$.

Furthermore, (20) gives a necessary and sufficient condition for ψ° to serve as the rental pseudomeasure for the given equilibrium $(E_1^\circ, E_2^\circ, \ldots)$. The economic interpretation of (20) is the same as in section 6: For each E_i°, ψ° lies between the highest bid and all the others. This generalizes (6.4) ~~(4) of section 6.~~

Our next result generalizes the "uniqueness" theorem of section 6 concerning decompositions of $(\psi_1, \psi_2, \ldots)$. In view of the theorems above, this also fixes the extent to which market equilibria, and maximizers of "social valuation", are unique.

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<u>Theorem</u>: Let $(\psi_1, \psi_2, \ldots)$ be a sequence of pseudomeasures on space (A, Σ), with extended Hahn decomposition $(E_1^\circ, E_2^\circ, \ldots)$; let $(E_1, E_2, \ldots)$ be another sequence of measurable sets that ~~which~~ partitions A (and has ~~having~~ the same length as $(E_1^\circ, E_2^\circ, \ldots)$).

Then $(E_1, E_2, \ldots)$ is also an extended Hahn decomposition iff

$$|\psi_i - \psi_j|\,(E_i^\circ \cap E_j) = 0 \qquad (6.9.26)\ \text{~~(26)~~}$$

for all $i, j = 1, 2, \ldots$.[19]

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<u>Proof</u>: Let $(E_1, E_2, \ldots)$ also be an extended Hahn decomposition, so that $(\psi_j - \psi_i)^-(E_j) = 0$, all $i, j = 1, 2, \ldots$. This can also be written: $(\psi_i - \psi_j)^+(E_j) = 0$. Also $(\psi_i - \psi_j)^-(E_i^\circ) = 0$ since $(E_1^\circ, E_2^\circ, \ldots)$ is a decomposition. Hence

$$|\psi_i - \psi_j|\,(E_i^\circ \cap E_j) = (\psi_i - \psi_j)^+(E_i^\circ \cap E_j) + (\psi_i - \psi_j)^-(E_i^\circ \cap E_j)$$

$$\leq (\psi_i - \psi_j)^+(E_j) + (\psi_i - \psi_j)^-(E_i^\circ) = 0,$$

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for all i, $j = 1, 2, \ldots$. This yields (26).

Conversely, let (26) hold. For all i, $k = 1, 2, \ldots,$ we obtain

$$(\psi_i - \psi_k)^-(E_k^\circ \cap E_i) \leq |\psi_i - \psi_k|(E_k^\circ \cap E_i) = 0, \qquad \text{(6.9.27)} \ \cancel{(27)}$$

from (26). Also, for all i, j, $k = 1, 2, \ldots,$ we obtain

$$(\psi_k - \psi_j)^-(E_k^\circ \cap E_i) \leq (\psi_k - \psi_j)^-(E_k^\circ) = 0, \qquad \text{(6.9.28)} \ \cancel{(28)}$$

since $(E_1^\circ, E_2^\circ, \ldots)$ is a decomposition. But also

$$(\psi_i - \psi_j)^- \leq (\psi_i - \psi_k)^- + (\psi_k - \psi_j)^- \qquad \text{(6.9.29)} \ \cancel{(29)}$$

is true for any three pseudomeasures ψ_i, ψ_j, ψ_k. (This follows from the minimizing property of the Jordan ~~decomposition)~~ form of a pseudomeasure.)
From (27), ~~(28), and~~ (29), we obtain

$$(\psi_i - \psi_j)^-(E_k^\circ \cap E_i) = 0, \qquad \text{(6.9.30)} \ \cancel{(30)}$$

for all i, j, $k = 1, 2, \ldots$. Finally, adding (30) over all k, we obtain

$$(\psi_i - \psi_j)^-(E_i) = 0,$$

all i, $j = 1, 2, \ldots,$ so that $(E_1, E_2, \ldots)$ is indeed another extended Hahn decomposition. |||

Condition (26) is a direct generalization of ~~(14) of section 6.~~ (6.14) The economic interpretation of ~~(14)~~ which ~~discussed in that section~~ carries over to (26).

Our final result generalizes the existence theorem of section 6. We have saved this for last because it is the only result ~~which~~ that demands just a finite number of agents. Every other result generalizes to the countable case. Indeed, the counterexample already given in section 6 shows that a countably infinite number of pseudomeasures may not have an extended Hahn decomposition.

Theorem: Any n-tuple of pseudomeasures, $(\psi_1,\ldots,\psi_n)$, on space (A,Σ) has an extended Hahn decomposition.

Proof: Proof is By induction on n. First, for $n = 2$, let (P,N) be a Hahn decomposition for the pseudomeasure $\psi_1 - \psi_2$. Then $(\psi_1 - \psi_2)^-(P) = 0$, and also

$$(\psi_2 - \psi_1)^-(N) = (\psi_1 - \psi_2)^+(N) = 0,$$

so that (P,N) is also an extended Hahn decomposition for the ~~part~~ pair (ψ_1, ψ_2).

Next, assuming the statement holds for n-1, we ~~shall~~ prove it for n. We have, then, a measurable n-1-tuple, $(E_1,\ldots,E_{n-1})$, ~~which~~ that partitions A, and for which

$$(\psi_i - \psi_j)^-(E_i) = 0, \tag{6.9.31}$$

for all $i, j = 1,\ldots,n-1$.

For each $i = 1,\ldots,n-1$, let (P_i, N_i) be a Hahn decomposition for $\psi_i - \psi_n$, and define:

$$E_i^\circ = E_i \cap P_i, \tag{6.9.32}$$

for $i = 1,\ldots,n-1$, and

$$E_n^\circ = (E_1 \cap N_1) \cup (E_2 \cap N_2) \cup \cdots \cup (E_{n-1} \cap N_{n-1}). \qquad (6.9.33)\ \cancel{(33)}$$

We ~~shall~~ prove that $(E_1^\circ,\ldots,E_n^\circ)$ is an extended Hahn decomposition for $(\psi_1,\ldots,\psi_n)$. It clearly partitions A.

For $i \neq n$, $j \neq n$, we have

$$(\psi_i - \psi_j)^-(E_i^\circ) \leq (\psi_i - \psi_j)^-(E_i) = 0,$$

~~and~~

$$(\psi_i - \psi_n)^-(E_i^\circ) \leq (\psi_i - \psi_n)^-(P_i) = 0,$$

both from (32).

It only remains to show that

$$(\psi_n - \psi_i)^-(E_n^\circ) = 0, \qquad (6.9.34)\ \cancel{(34)}$$

for all $i \neq n$.

For any $i, j = 1,\ldots,n-1$ we have

$$(\psi_j - \psi_i)^-(E_j \cap N_j) = 0, \qquad (6.9.35)\ \cancel{(35)}$$

from (31). Also

$$(\psi_n - \psi_j)^-(E_j \cap N_j) \leq (\psi_n - \psi_j)^-(N_j) = 0. \qquad (6.9.36)\ \cancel{(36)}$$

(35), (36), and (29) imply that

$$(\psi_n - \psi_i)^-(E_j \cap N_j) = 0. \qquad (6.9.37)\ \cancel{(37)}$$

Adding (37) over all $j = 1,\ldots,n-1$, and noting (33), we obtain (34). This shows that $(E_1^\circ,\ldots,E_n^\circ)$ is indeed a decomposition for $(\psi_1,\ldots,\psi_n)$, and completes the induction and the proof. ~~///~~ $\square$

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Appendix: The Vector Lattice of Pseudomeasures

We ~~shall briefly~~ indicate some algebraic consequences of the results of the last section. ~~These are relegated~~ in to an appendix because they are not applied in this book, but they are of interest for the further mathematical development of pseudomeasure theory.

Consider the vector space of all pseudomeasures, Ψ, on measurable space (A,Σ). Let $\{\psi_1,\ldots,\psi_n\}$ be a finite non-empty subset of Ψ, and suppose that ψ_o has the following properties:

(i) $\psi_o \geq \psi_i$, for all $i = 1,\ldots,n$ ("$\geq$" refers to narrow order throughout this section); and

(ii) for any ψ, if $\psi \geq \psi_i$ for all $i = 1,\ldots,n$, then $\psi \geq \psi_o$.

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Definition: Such a pseudomeasure ψ_o (if it exists) will be called the supremum of $\{\psi_1,\ldots,\psi_n\}$. Similarly, a ψ_o satisfying (i) and (ii), but with "$\geq$" replaced by "$\leq$", will be called the infimum of $\{\psi_1,\ldots,\psi_n\}$. ~~We shall make~~ the ~~usual~~ abbreviations, sup and inf, are used for these operations.

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Let $\{\psi_1,\ldots,\psi_n\}$ has at most one supremum. For, if ψ_o', ψ_o'' both satisfy (i) and (ii), then $\psi_o' \geq \psi_o'' \geq \psi_o'$. Hence $\psi_o' = \psi_o''$, since narrow order is antisymmetric. Similarly, it has at most one infimum.

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Theorem: Any non-empty finite set of pseudomeasures, $\{\psi_1,\ldots,\psi_n\}$, on space (A,Σ), has a supremum and an infimum. The supremum is, ~~in fact,~~ given by

$$\psi_o = (\psi_1 | E_1^o) \oplus \ldots \oplus (\psi_n | E_n^o), \qquad \text{(6.9.38)} \; \cancel{(38)}$$

where $(E_1^o, \ldots, E_n^o)$ is any extended Hahn decomposition of $(\psi_1, \ldots, \psi_n)$. The infimum is given by

$$-\sup\{-\psi_1, \ldots, -\psi_n\}. \qquad \text{(6.9.39)} \; \cancel{(39)}$$

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Proof: We know that an extended Hahn decomposition exists, so there is at least one pseudomeasure of the form (38). We also know that (38) is greatest under narrow ordering in the set Ψ_E of all pseudomeasures of the same form, with $(E_1, \ldots, E_n)$ in place of $(E_1^o, \ldots, E_n^o)$; ~~—~~ where $(E_1, \ldots, E_n)$ is a measurable n-tuple that ~~which~~ partitions A.

In particular, take the n-tuple $(\emptyset, \ldots, A, \ldots, \emptyset)$, with the universe set in place i, the empty set everywhere else. The pseudomeasure of form (38) corresponding to this is simply ψ_i. Hence $\psi_o \geq \psi_i$, ~~all~~ $i = 1, \ldots, n$.

Next, suppose $\psi \geq \psi_i$, ~~all~~ $i = 1, \ldots, n$, for some ψ. This implies $(\psi - \psi_i)^-(E_i^o) = 0$, $i = 1, \ldots, n$. Now ψ_o coincides with ψ_i when both are restricted to E_i^o. Hence $(\psi - \psi_o)^-(E_i^o) = 0$ for all i. Adding over i, we obtain $(\psi - \psi_o)^-(A) = 0$; i.e., ~~— that is,~~ $\psi \geq \psi_o$. This proves that ψ_o is indeed the supremum.

Now ~~Let~~ let ψ_{oo} abbreviate (39). We then have $-\psi_{oo} \geq -\psi_i$, all i, and, if $\psi \geq -\psi_i$, all i, then $\psi \geq -\psi_{oo}$. These are the same as

$\psi_{oo} \leq \psi_i$, all *i*; and, if $-\psi \leq \psi_i$, all *i*, then $-\psi \leq \psi_{oo}$, which in turn is the same as (i) and (ii) with signs reversed. Hence ψ_{oo} is the infimum.

A number of corollaries implicit in this theorem illuminate the concepts of the preceding ~~section.~~ discussion. First, the supremum is defined for a set, while the expression (38) is in terms of a particular ordering $(\psi_1, \ldots, \psi_n)$ of the elements of this set. A moment's reflection shows, however, that the ordering is irrelevant. Indeed, a permutation of $(\psi_1, \ldots, \psi_n)$ leads to a corresponding permutation of the decomposition $(E_1^{\circ}, \ldots, E_n^{\circ})$. This leads merely to a change in the order of the summands in (38), and ~~this~~ yields the same pseudomeasure.

Second, even though there may be many extended Hahn decompositions for $(\psi_1, \ldots, \psi_n)$, these all must yield the same pseudomeasure (38), since the supremum is unique.

Sup $\{\psi_1, \ldots, \psi_n\}$ is exactly what was referred to above ~~in the preceding section~~ as the social valuation of the real-estate equilibrium. This provides a concrete interpretation.

The fact that the sup and inf always exist means that Ψ is not only a vector space, but a lattice (with respect to narrow ordering).[20]

We conclude with some (fairly difficult) exercises:

(i) Show that, for any two pseudomeasures $\psi_1, \psi_2 \in \Psi$,

$$\sup\{\psi_1, \psi_2\} = \frac{1}{2}\left(\psi_1 + \psi_2 + |\psi_1 - \psi_2|\right),$$

~~and~~

$$\underline{\inf}\{\psi_1, \psi_2\} = \tfrac{1}{2}\left(\psi_1 + \psi_2 - |\psi_1 - \psi_2|\right).$$

(Here the total variation $|\psi_1 - \psi_2|$ is ~~of course~~ a measure, which may be identified, as usual, with the pseudomeasure $(|\psi_1 - \psi_2|, 0)$.)

(Hint: the Hahn decomposition of $\psi_1 - \psi_2$ is the same as the extended Hahn decomposition of (ψ_1, ψ_2)).

2. ~~(ii)~~ If ψ_1, ψ_2 are measures, show that these operations coincide with the ordinary sup and inf of measures (defined in ~~chapter 3,~~ section 3.1).

3. ~~(iii)~~ Show that Ψ under narrow order is in fact a distributive lattice; ~~that is~~ i.e., for any three pseudomeasures ψ_1, ψ_2, ψ_3,

$$\inf\left\{\psi_1, \sup\{\psi_2, \psi_3\}\right\} = \sup\left\{\inf\{\psi_1, \psi_2\}, \inf\{\psi_1, \psi_3\}\right\},$$

and a similar equality holds for "inf" and "sup" interchanged. (Hint: First do the special case $\psi_1 = 0$; take a Hahn decomposition of $\psi_2 - \psi_3$ and do each half separately).

4. ~~(iv)~~ Under standard ordering, show that Ψ (partitioned into indifference classes) is not a lattice, unless Σ is a finite ~~sigma~~ σ-field. In fact, show that ψ_1, ψ_2 have a least upper bound under standard order iff they are comparable under standard order (i.e., iff $\psi_1 - \psi_2$ is a signed measure).

FOOTNOTES - CHAPTER 6

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[1] If $\lambda_2 = 0$, ~~then~~ (4) states that b is a distribution function for λ_1, in the wide sense. This is ~~no~~ so because we have defined distribution functions to be continuous from below, which in this case means: from the past.

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540 [2] The real-estate market is especially rich in having diverse agents with interests in the same parcel: owners, developers, builders, tenants, holders of easements, government agencies, ~~etc.~~ See R. Turvey, The Economics of Real Property (Allen and Unwin, London, 1957), ~~pages~~ pp. 4–5; and W. L. C. Wheaton, "Public and Private Agents of Change in Urban Expansion", in Explorations into Urban Structure, M. M. Webber et al., ~~et al.~~ (Univ. ~~ersity of~~ Pa. ~~Pennsylvania~~ Press, Philadelphia, 1964), ~~pages~~ pp. 171–175.

542 ~~3.~~ The "separation of ownership and control" as a social problem was first broached in connection with corporations. See A. A. Berle, Jr., and G. C. Means, The Modern Corporation and Private Property (Commerce Clearing House, New York, 1932). Our analysis indicates this separation is a universal phenomenon. ~~It is true that,~~ in the corporate case, there are special institutional obstacles to having the assets revert to the control of their legal owners. But compare this situation with the case of self-perpetuating boards of trustees or church hierarchies, where there are no legal owners at all!

3 2a cf. "Bailment", in ~~pages 179-180 of~~ H. C. Black, Black's Law Dictionary, rev. 4th ed. (West Publishing ~~Company~~, St. Paul, ~~Minn., revised 4th edition~~, 1968), pp. 179–80. None of the meanings described in this article is quite identical with the one we are using.

~~¶ 5. See note 3.~~

(543) ¶ 5 [4]See G. S. Becker, Human Capital (Nat~~ional~~ Bur~~eau of~~ Econ~~omic~~ Res~~earch~~, New York, 1964), ch~~apter~~ 2, for an extended analysis.

(544) ¶ 6 [5]G. S. Becker, The Economics of Discrimination, 2nd ed. (Univ~~ersity~~ ~~of~~ Chicago Press, ~~Chicago,~~ 1971), ~~page~~ p. 15.

~~(547) ¶ 8. [6]Incidentally, the converse of this statement is false: Not all linear functions on the space of bounded signed measures can be represented as integrals.~~

(548) ¶ 7. [7]Strictly speaking, (1) is not a special case of (2), because the function g just given is not bounded, and ~~because~~ (2) is valid only for the subset M'. M-prime

(553) ¶ 8. [8]One could insert an intermediate step here, deflating by a price index P(t) to get measurements in "constant dollars." For consistency, one then ~~has to~~ must subtract the "rate of inflation," DP(t)/P(t), from (4) to get the "real" rate of discount. One then proceeds as above, with "real" or "constant" values in place of "current" values.

(568) ¶ 9. [9]J. R. P. Friedmann, The Spatial Structure of Economic Development in the Tennessee Valley (Univ~~ersity of~~ Chicago Press, Chicago, 1955), ~~pages~~ pp. 35, 42–43.

588 # 15. ~~17~~ ~~15~~ This is a special case of what we shall call Thünen systems, which are examined in great detail in chapter 8.

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588 # 16. ~~18~~ ~~16~~ W. Alonso, *Location and Land Use* (Harvard Univ~~ersity~~ Press, Cambridge, 1964). The heart of the theory is in chapters 3, 4, 5 and appendices A and B. W. Alonso, "A Reformulation of Classical Location Theory and Its Relation to Rent Theory", *Regi~~onal~~ Science Associ~~ation~~ Pape~~rs~~*, 19 (1967): ~~23-44, 1967, at~~ page 41.

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592 # 17. ~~19~~ ~~17~~ Shape is considered by Alonso in Appendix B of his book. We discuss this below.

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600 # 18. There is one minor exception to the statement that this section generalizes Section 6.6. The existence theorem there is valid for *arbitrary* signed measures, not merely ~~sigma~~ σ-finite ones. Since pseudomeasures generalize only *~~sigma~~ σ-finite* signed measures, this theorem is not completely encompassed in the present results. The existence theorem is also the exceptional theorem mentioned above.

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617 # 19. Recall that, for any pseudomeasure ψ, $|\psi|$ is a *measure* called the *total variation* of ψ, and is equal to $\psi^+ + \psi^-$.

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623 # 20. G. Birkhoff, *Lattice Theory*, Vol. 25, 3rd ed. (Ame~~rican~~ Mathe~~matical~~ Soc~~iety~~ Colloq~~uium~~ Publ~~ications~~, ~~vol. 25,~~ Providence, R.I., ~~3rd ed.,~~ 1967).

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(568) ¶ 10 ~~10~~ The premium is known as "plottage value". A. M. Weimer and H. Hoyt, Principles of Real Estate, 4th ed. (Ronald, New York, ~~4th ed~~, 1960), ~~pages~~ pp. 285–286.

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(569) ¶ 11 ~~11~~ C. P. Barnes, "Economics of the Long-lot Farm," Geographical Review, 25: 298–301, (1935); M. Chisholm, Rural Settlement and Land Use (Hutchinson, London, 1962), ~~page~~ p. 156.

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(577) ¶ 12 ~~12~~ If $\mu_1, \ldots, \mu_n$ are all sigma-finite, there is an alternative – but less straightforward – proof based on the Radon-Nikodym theorem. Cf. the proof of Theorem 2 of L. E. Dubins and E. H. Spanier, "How to Cut a Cake Fairly", American Mathematical Monthly, 68:1–17, (1961), reprinted in Readings in Mathematical Economics, P. Newman (ed.) (Johns Hopkins Press, Baltimore, 1968), vol. 1.

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(584) ¶ 13 ~~13~~ The first use of this concept is by R. J. Aumann, "Markets with a Continuum of Traders," Econometrica, 32:39–50, (January-April, 1964), reprinted in Readings ~~in Mathematical Economics~~, ~~P.~~ Newman (ed.) ~~(Johns Hopkins University Press, Baltimore, 1968), vol. 1.~~ An extensive literature has grown up since then.

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(585) ¶ 14 ~~14~~ Econometrica, 32 (1964): 39. ~~January-April, 1964.~~
