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3. THE COMPARISON OF INFINITE MEASURES

This chapter develops the theory of pseudomeasures. These are extensions of signed measures ~~which~~ that enable one to carry out, for example, the operation of subtraction even for infinite measures. ~~It turns out that~~ much of standard measure theory generalizes to pseudomeasures, and ~~that~~ many theorems can be stated without qualifying conditions as to finiteness, integrability, etc. Thus the theory should ~~hold~~ interest even ~~for~~ "pure" mathematicians.

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The theory also has numerous applications. First, it enables one to "net" freely, even when both "grosses" are infinite. The subtraction of consumption from production has already been discussed. Another example is migration: One would like to get net migration by subtracting gross out-migration from in-migration, even when the latter two measures are infinite. (This might occur, e.g., ~~for example,~~ on an infinite plane, or with an infinite time-horizon). ~~And,~~ in general, it enables one to perform arithmetical accounting operations freely on measures, without worrying about the appearance of the meaningless expression $\infty - \infty$.

Second, it allows one to compare different "infinite utility streams" such as arise in the evaluation of economic development programs. The "overtaking" and similar criteria ~~which have been~~ developed to deal with these problems find their natural place within the theory, and emerge as special cases of a general approach.

Even more generally, pseudomeasures ~~turn out~~ are in many cases ~~to be~~ a natural way of representing preference orderings. i.e., ~~That is,~~ instead of representing preferences by real-valued utility functions, one uses pseudomeasure-valued utility functions, with various natural orderings on the space of pseudomeasures. This arises for infinite-horizon development programs, for problems of location theory on the infinite plane, and for preferences among uncertain situations.

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3.1. Jordan Decomposition Theory

The formal development of pseudomeasure theory goes through two stages. The first ~~stage~~ involves a generalization of the concept of Jordan decomposition. This operation, which applies to any pair of measures, has an interesting and elegant theory by itself. In ~~the present~~ this section we ~~shall~~ develop only that portion of the theory which lends directly to pseudomeasures, or ~~which~~ has direct applications elsewhere in this book. Some other results will be presented as exercises (which are generally fairly difficult to prove).

Pseudomeasures per se arise from the application of the Jordan decomposition to sigma-finite measures. This enables one to define algebraic operations, integration, and ordering relations in a manner ~~which~~ that is fruitful for applications, and also of considerable mathematical interest in itself.

Let $(\underline{A},\Sigma)$ be a measurable space. All sets referred to below are assumed to belong to Σ, and all measures and other set functions are assumed to have Σ as their domain.

Definition: Let (μ,ν) be an ordered pair of measures; the upper variation of (μ,ν) is the set function λ^+ given by

$$\lambda^+(E) = \sup\{\mu(F) - \nu(F) \mid F \subseteq E,\ \nu(F) < \infty\}, \qquad \text{(3.1.1)} \ \cancel{(1)}$$

for all $E \in \Sigma$.

That is, to find $\lambda^+(E)$, form the difference $\mu(F) - \nu(F)$ for each measurable subset F of E for which $\nu(F)$ is finite, and then take the supremum over these numbers. Similarly, consider the following.

Definition: The lower variation of (μ,ν) is the set function λ^- given by

$$\lambda^-(E) = \sup\{\nu(F) - \mu(F) \mid F \subseteq E,\ \mu(F) < \infty\}, \qquad \text{(3.1.2)} \ \cancel{(2)}$$

for all E element $\in \Sigma$.

Definition: The Jordan decomposition of the ordered pair of measures (μ,ν) is the ordered pair of set functions (λ^+,λ^-) given by (1), (2).[1]

Theorem: The set functions λ^+, λ^- are actually measures.

Proof: ~~It is clear that~~ λ^+ and λ^- are non-negative (since $\emptyset \subseteq E$, for all $E \in \Sigma$), and ~~that~~ $\lambda^+(\emptyset) = \lambda^-(\emptyset) = 0$. It remains only to prove countable additivity.

Let (E_n), $n = 1, 2, \ldots$, be a countable measurable packing, with $E = \cup\{E_n\}$. Let F satisfy: $F \subseteq E$, $F \in \Sigma$, $\nu(F) < \infty$; then $F \cap E_n$ fulfils the same conditions with E_n in place of E, for all $n = 1, 2, \ldots$. Hence

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$$\lambda^+(E_1) + \lambda^+(E_2) + \cdots \geq [\mu(F\cap E_1) - \nu(F\cap E_1)] + [\mu(F\cap E_2) - \nu(F\cap E_2)] + \cdots$$

$$= \mu(F) - \nu(F). \qquad \text{(3.1.3)} \; \cancel{(3)}$$

Taking the supremum over all such sets F, we obtain

$$\lambda^+(E_1) + \lambda^+(E_2) + \cdots \geq \lambda^+(E). \qquad \text{(3.1.4)} \; \cancel{(4)}$$

It remains to establish the opposite inequality. If $\lambda^+(E_n) = \infty$ for any n, then $\lambda^+(E) = \infty$, since λ^+ is clearly monotone non-decreasing; in this case we get equality in (4). The remaining case is where $\lambda^+(E_n)$ is finite for all n. Choose a real number $\varepsilon > 0$, and, for each $n = 1, 2, \ldots$, choose $F_n \subseteq E_n$ such that

$$\mu(F_n) - \nu(F_n) \geq \lambda^+(E_n) - \varepsilon \cdot 2^{-n}. \qquad \text{(3.1.5)} \; \cancel{(5)}$$

Noting that $(F_1 \cup \cdots \cup F_N) \subseteq E$, and adding (5) over $n = 1, \ldots, N$, we obtain

$$\lambda^+(E) \geq [\mu(F_1) - \nu(F_1)] + \cdots + [\mu(F_N) - \nu(F_N)]$$

$$\geq \lambda^+(E_1) + \cdots + \lambda^+(E_N) - \varepsilon(2^{-1} + \cdots + 2^{-N}). \qquad \text{(3.1.6)} \; \cancel{(6)}$$

Letting $N \to \infty$ in (6), we obtain

$$\lambda^+(E) \geq -\varepsilon + \lambda^+(E_1) + \lambda^+(E_2) + \cdots.$$

Since $\varepsilon > 0$ is arbitrary, we obtain (4) with inequality sign reversed. Hence λ^+ is countably additive. By symmetry, so is λ^-. ∎

How does this operation compare with the ordinary Jordan decomposition of a signed measure λ? We know that λ can be expressed as the difference of two measures, say $\lambda = \lambda_1 - \lambda_2$, where λ_1 or λ_2 is finite. Let (λ^+,λ^-) be the (generalized) Jordan decomposition of the pair (λ_1,λ_2). It is then easily verified that λ^+, λ^- coincide with the ordinary upper and lower variations of λ, respectively. Note that the operation above is well-defined even if μ and ν are both infinite. In this sense it represents a true generalization of the ordinary Jordan decomposition.

~~Let us~~ write $J(\mu,\nu)$ for the Jordan decomposition of (μ,ν). Let ρ_1, ρ_2 be set functions. We write $\rho_1 \leq \rho_2$ to indicate that $\rho_1(E) \leq \rho_2(E)$ for all sets $E \in \Sigma$. Then we have the following.

Theorem: Let $(\lambda^+,\lambda^-) = J(\mu,\nu)$; then

$$\lambda^+ \leq \mu \text{ and } \lambda^- \leq \nu. \qquad (3.1.7)\ \ \text{~~(7)~~}$$

Proof: Choose $E \in \Sigma$. For any $F \subseteq E$ with $\nu(F) < \infty$, we have

$$\mu(E) \geq \mu(F) \geq \mu(F) - \nu(F).$$

Taking the supremum over all such F, we obtain $\mu(E) \geq \lambda^+(E)$. Thus $\mu \geq \lambda^+$. The proof that $\nu \geq \lambda^-$ is similar. ~~///~~ □

Thus J is a "shrinking" operation. We ~~shall~~ see below that J in effect removes the common part of μ, ν from each of them.

§9.3

Let μ, ν be measures, with $\mu \geq \nu$. We want to define the operation of subtracting ν from μ in a reasonable way. One's first impulse is to take $\mu(E) - \nu(E)$, but this introduces the meaningless operation $\infty - \infty$ if μ and ν are both infinite measures.

Definition: Let $\mu \geq \nu$ be measures. $\mu - \nu$ is defined as the upper variation of the pair (μ, ν).

Note that $\mu - \nu$ is only defined for the case $\mu \geq \nu$. One easily sees, incidentally, that the lower variation of (μ, ν) is 0, the identically zero measure.

The following theorem shows that "minus" has at least some of the properties of ordinary subtraction.

Theorem: (i) Let $\mu \geq \nu$ be measures; then

$$\mu = (\mu - \nu) + \nu. \tag{3.1.8}$$

(ii) Let μ, ν, θ be measures, with $\mu = \nu + \theta$; then

$$\theta \geq (\mu - \nu). \tag{3.1.9}$$

(If ν is ~~sigma-finite~~ σ-finite, then (9) is an equality).

Proof: (i) If $\nu(E) = \infty$, then $\mu(E) = \infty$, and (8) is satisfied at E. If $\nu(E) < \infty$, then, for any $F \subseteq E$,

$$\mu(E) - \nu(E) = [\mu(F) - \nu(F)] + [\mu(E \setminus F) - \nu(E \setminus F)] \geq \mu(F) - \nu(F),$$

so that $\mu(F) - \nu(F)$ attains its supremum at $F = E$; hence

$$(\mu - \nu)(E) = \mu(E) - \nu(E), \qquad (3.1.10) \ \cancel{(10)}$$

which again verifies (8) at E; this proves ~~part~~ (i).

(ii) Choose $E \in \Sigma$. For any $F \subseteq E$ such that $\nu(F) < \infty$, we have

$$\theta(E) \geq \theta(F) = \mu(F) - \nu(F).$$

Taking the supremum over all such F, we obtain (9). Finally, let ν be ~~sigma~~-finite, so that there is a measurable partition (A_n), $n = 1, 2, \ldots$, of A such that $\nu(A_n) < \infty$, all n. Then

$$\theta(E \cap A_n) = \mu(E \cap A_n) - \nu(E \cap A_n)$$
$$= (\mu - \nu)(E \cap A_n), \qquad (3.1.11) \ \cancel{(11)}$$

from (10). Summing (11) over n, we obtain (9) with equality. ///

Note that the inequality (9) is sometimes strict; e.g., let A consist of one point, and let $\mu(A) = \nu(A) = \infty$, $\theta(A) = 1$.

Exercises:

~~(i)~~ 1 Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ be measures. Show that

$$(\mu_1 - \mu_n) = (\mu_1 - \mu_2) + (\mu_2 - \mu_3) + \ldots + (\mu_{n-1} - \mu_n).$$

(Hint: Prove for $n = 3$ and use induction; consider separately the two cases $(\mu_{n-1} - \mu_n)(E) = \infty, < \infty$).

~~(ii)~~ 2 Let $\mu \geq \nu \geq \theta$ be measures. Show that

$$(\mu - \nu) = (\mu - \theta) - (\nu - \theta).$$

(Hint: Use the result of the preceding exercise for a start).

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We now define two further operations on a pair of measures (μ,ν). Consider measures θ satisfying $\theta \le \mu$, $\theta \le \nu$. Is there a largest among them? That is, is there a measure $\tilde{\theta}$ satisfying these conditions, and $\geq \theta$ for any θ satisfying them? There is, and it is known as the infimum of μ and ν, written $\underline{\inf}(\mu,\nu)$. ~~In fact,~~ one can give an explicit formula for this measure:

$$\underline{\inf}(\mu,\nu)(E) = \underline{\inf}\{\mu(F) + \nu(E\setminus F) \mid F \subseteq E\}, \tag{3.1.12}$$

for all $E \in \Sigma$.

Note the distinction between the two "infs" in (12)! The one on the right is the ordinary infimum of a set of extended real numbers, ~~namely,~~ viz. $\mu(F) + \nu(E\setminus F)$ for all measurable subsets F of E.

Similarly, the supremum of μ and ν, written $\underline{\sup}(\mu,\nu)$, is the smallest measure $\geq \mu, \nu$. The formula for this is the same as (12), with (ordinary) "sups" in place of "inf" on the right-hand side.[2]

9.9.4

Theorem: Let μ, ν be measures, and let $(\lambda^+,\lambda^-) = \underline{J}(\mu,\nu)$; then

$$\mu + \lambda^- = \nu + \lambda^+ = \underline{\sup}(\mu,\nu). \tag{3.1.13}$$

Proof: First, we prove the right-hand equality in (13). Choose $E \in \Sigma$. If $\nu(E) = \infty$, this equality is ~~clearly~~ satisfied at E. If $\nu(E) < \infty$, then

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$$\begin{aligned}\nu(E) + \lambda^+(E) &= \nu(E) + \underline{\sup}\{\mu(F) - \nu(F) \mid F \subseteq E\}\\ &= \underline{\sup}\{\mu(F) + [\nu(E) - \nu(F)] \mid F \subseteq E\}\\ &= \underline{\sup}\{\mu(F) + \nu(E\setminus F) \mid F \subseteq E\} = \underline{\sup}(\mu,\nu)(E),\end{aligned}$$

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so the right-hand equality again holds at $\underline{E}$. In a similar manner, with μ in place of ν, we prove that

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$$\mu + \lambda^{-} = \underline{\sup}(\mu,\nu),$$

which establishes (13).

This result has several applications. As a first, we show that the Jordan decomposition operator is idempotent. That is, since $\underline{J}(\mu,\nu)$ is an ordered pair of measures, we may apply the $\underline{J}$ operator again; but ~~it turns out that~~ nothing new arises:

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$$\underline{J}^2(\mu,\nu) = \underline{J}(\underline{J}(\mu,\nu)) = \underline{J}(\mu,\nu).$$

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Theorem: The Jordan decomposition operator satisfies $\underline{J}^2 = \underline{J}$.

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Proof: Let μ_0, ν_0 be measures, let $(\mu_1,\nu_1) = \underline{J}(\mu_0,\nu_0)$, and let $(\mu_2,\nu_2) = \underline{J}(\mu_1,\nu_1)$. We must show that $\mu_2 = \mu_1$, $\nu_2 = \nu_1$. It suffices~~, in fact,~~ to show that $\mu_2 \geq \mu_1$, $\nu_2 \geq \nu_1$, since the opposite inequalities are already known, by (7) ~~above.~~ Choose $\underline{E} \in \Sigma$. We then have

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$\mu_2 \geq \mu_1$

$$\mu_1(\underline{E}) = \underline{\sup}\{\mu_1(\underline{F}) - \nu_1(\underline{F}) \mid \underline{F} \subseteq \underline{E},\ \nu_0(\underline{F}) < \infty\}. \qquad (3.1.14)\ (14)$$

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To see this, note that

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$$\mu_1(\underline{F}) + \nu_0(\underline{F}) = \mu_0(\underline{F}) + \nu_1(\underline{F}) \qquad (3.1.15)\ \cancel{(15)}$$

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by (13) ~~above.~~ Also $\nu_0(\underline{F}) < \infty$ for the sets $\underline{F}$ in (14), and $\nu_1(\underline{F}) \leq \nu_0(\underline{F})$ by (7). Hence we may transpose the ν-terms in

(15) to obtain

$$\mu_1(F) - \nu_1(F) = \mu_\Theta(F) - \nu_\Theta(F), \qquad \text{zero}$$

from which (14) follows. (14) in turn implies that

$$\mu_1(E) \le \sup\{\mu_1(F) - \nu_1(F) \mid F \subseteq E,\ \nu_1(F) < \infty\}. \qquad \text{(3.1.16)}\ \cancel{(16)}$$

For the set of numbers over which the sup is taken in (16) is at least as large as the set in (14), since $\nu_1 \le \nu_\Theta$. But (16) states that $\mu_1(E) \le \mu_2(E)$. Hence $\mu_1 = \mu_2$. The proof that $\nu_1 = \nu_2$ is similar. ||| zero

The following important result is a second application.

Theorem: Let $\mu_1, \nu_1, \mu_2, \nu_2$ be measures such that

$$J(\mu_1,\nu_1) = J(\mu_2,\nu_2); \qquad \text{(3.1.17)}\ \cancel{(17)}$$

then

$$\mu_1 + \nu_2 = \nu_1 + \mu_2. \qquad \text{(3.1.18)}\ \cancel{(18)}$$

Proof: By contradiction. Suppose ~~that~~ (λ^+,λ^-) is the common Jordan decomposition of (μ_1,ν_1) and (μ_2,ν_2), and let (18) be false, so that there is an $E \in \Sigma$ for which, say,

$$\mu_1(E) + \nu_2(E) < \nu_1(E) + \mu_2(E). \qquad \text{(3.1.19)}\ \cancel{(19)}$$

Hence $\mu_1(E) < \infty$, so that $\lambda^+(E) < \infty$; also $\nu_2(E) < \infty$, so that $\lambda^-(E) < \infty$. Now we have

$$\mu_i(E) + \lambda^-(E) = \nu_i(E) + \lambda^+(E),$$

$\underline{i} = 1, 2$, by (13). Adding, we obtain

$$\mu_1(\underline{E}) + \lambda^-(\underline{E}) + \nu_2(\underline{E}) + \lambda^+(\underline{E}) = \nu_1(\underline{E}) + \lambda^+(\underline{E}) + \mu_2(\underline{E}) + \lambda^-(\underline{E}).$$

The λ-terms, being finite, drop out, and we are left with a contradiction of (19). If the inequality in (19) is reversed, the same argument again leads to a contradiction. Hence (18) is true. |||

It will turn out, under ~~sigma~~ σ-finiteness assumptions, that (17) and (18) are actually equivalent, a basic result for pseudomeasures.

9.9.5

Theorem: Let μ, ν be measures, let $(\lambda^+,\lambda^-) = \underline{J}(\mu,\nu)$, and let $\theta = \underline{\inf}(\mu,\nu)$; then

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$$\lambda^+ = \mu - \theta, \quad \lambda^- = \nu - \theta, \qquad (3.1.20)\ \cancel{(20)}$$

~~and~~

$$\mu = \lambda^+ + \theta, \quad \nu = \lambda^- + \theta. \qquad (3.1.21)\ \cancel{(21)}$$

Proof: $(\mu - \theta)$ is the upper variation of (μ,θ), while λ^+ is the upper variation of (μ,ν). Since $\nu \geq \theta$, and the upper variation is a non-increasing function of the right-hand component of the pair $(\mu,\cdot)$, it follows that

$$\lambda^+ \leq (\mu - \theta). \qquad (3.1.22)\ \cancel{(22)}$$

To prove the converse inequality, choose $\underline{E} \in \Sigma$; for any $\underline{F} \subseteq \underline{E}$ such that $\theta(\underline{F}) < \infty$, and for any finite $\underline{\varepsilon} > 0$, there is a measurable $\underline{G} \subseteq \underline{F}$ such that

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$$\theta(F) \geq \mu(F \backslash G) + \nu(G) - \varepsilon, \qquad (3.1.23)\ \cancel{(23)}$$

by definition (12). We then have

$$\mu(F) - \theta(F) \leq \mu(G) - \nu(G) + \varepsilon \leq \lambda^{+}(E) + \varepsilon. \qquad (3.1.24)\ \cancel{(24)}$$

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The left inequality in (24) follows from (23); the right inequality is true by definition of λ^{+}, on noting that $G \subseteq E$ and $\nu(G) \leq \theta(F) + \varepsilon < \infty$, by (23).

Taking the supremum in (24) over such sets F, we obtain

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$$(\mu - \theta)(E) \leq \lambda^{+}(E) + \varepsilon.$$

Since ε is arbitrary, we obtain (22) with inequality sign reversed. This establishes $\lambda^{+} = (\mu - \theta)$. The proof that $\lambda^{-} = (\nu - \theta)$ is similar. Hence (20) is established.

Finally, (21) follows from (20), e.g.,

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$$\lambda^{+} + \theta = (\mu - \theta) + \theta = \mu,$$

by (8). ||| □ ■

The results (20) are intuitively appealing. Inf(μ,ν) may be thought of as the mass distribution ~~which~~ That μ and ν share in common. (20) then states that the Jordan decomposition operator subtracts out this common part from μ and ν, respectively. (One should not jump to the conclusion, however, that λ^{+}, λ^{-} have nothing in common: inf$(\lambda^{+},\lambda^{-})$ is not always 0. See below).

It follows from (21) and (9) that

$$\theta \geq (\mu - \lambda^+) \boxed{,} \quad \theta \geq (\nu - \lambda^-), \qquad \text{(3.1.25)} \ \cancel{(25)}$$

with equality if λ^+ or λ^- is ~~sigma~~ σ-finite, respectively. ~~(25)~~ ((25) in general, cannot be strengthened to equality: Take $A = \{a\}$, $\mu(A) = \infty$, $\nu(A) = 1$).

Results (21) furnish alternative proofs for two preceding theorems of importance: the idempotency of J, and the equality (18) for two pairs with the same Jordan decomposition. Taking the latter first, assume (17) with (λ^+, λ^-) the common decomposition, and let $\theta_i = \inf(\mu_i, \nu_i)$, $i = 1, 2$. Then

$$\mu_1 + \nu_2 = (\lambda^+ + \theta_1) + (\lambda^- + \theta_2) = (\lambda^- + \theta_1) + (\lambda^+ + \theta_2) = \nu_1 + \mu_2, \qquad \text{(3.1.26)} \ \cancel{(26)}$$

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by (21), which yields (18). As for idempotency, let $(\mu_1, \nu_1) = J(\mu_\theta, \nu_\theta)$ and $(\mu_2, \nu_2) = J(\mu_1, \nu_1)$. The hard thing to prove is that $\mu_2 \geq \mu_1$, $\nu_2 \geq \nu_1$; but, by (21), we have

zero

$$(\mu_1, \nu_1) = J(\mu_1 + \theta,\ \nu_1 + \theta), \qquad \text{(3.1.27)} \ \cancel{(27)}$$

where $\theta = \inf(\mu_\theta, \nu_\theta)$. Considering θ in (27) as a variable measure, ~~one easily~~ we verifies that upper and lower variations are both non-increasing functions of θ, so that, indeed, $\mu_1 \leq \mu_2$, $\nu_1 \leq \nu_2$, implying $J^2 = J$.

zero

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Exercises:

~~(i)~~ 1 Let $(\lambda^+,\lambda^-) = J(\mu,\nu)$. Show that

$$\lambda^+ = \sup(\mu,\nu) - \nu, \text{ that } \lambda^- = \sup(\mu,\nu) - \mu, \text{ and that}$$

$$\lambda^+ + \lambda^- = [\sup(\mu,\nu) - \inf(\mu,\nu)].$$

(Hint: start with (13)).

~~(ii)~~ 2 Let $J(\mu_1,\nu_1) = J(\mu_2,\nu_2)$. Show that the common value in (18) is

$$\sup(\mu_1,\nu_1) + \inf(\mu_2,\nu_2) = \inf(\mu_1,\nu_1) + \sup(\mu_2,\nu_2).$$

(Hint: this follows easily from (21) and (13)).

9.9.6

A final cluster of ideas centers on the concepts of Hahn decomposability and mutual singularity. These have already been discussed in connection with signed measures, and we repeat the definitions here for convenience.

Definition: An ordered pair of measures (μ,ν) is <u>Hahn decomposable</u> iff universe set A can be split into two measurable pieces, P and N, such that $\mu(E) \geq \nu(E)$ for all measurable $E \subseteq P$, and $\mu(F) \leq \nu(F)$ for all measurable $F \subseteq N$. The ordered pair (P,N) is a <u>Hahn decomposition</u> for (μ,ν).

Definition: An ordered pair of measures (μ,ν) is <u>mutually singular</u> iff A can be split into two measurable pieces, P and N, such that $\nu(P) = 0$ and $\mu(N) = 0$.

~~It is clear that~~ mutual singularity implies Hahn decomposability, since the pair (P,N) is a Hahn decomposition.

Theorem: Let μ, ν be measures, with $(\lambda^+,\lambda^-) = J(\mu,\nu)$. Each of the following conditions implies the other two:

(i) (μ,ν) is Hahn decomposable;

(ii) (λ^+,λ^-) is Hahn decomposable;

(iii) (λ^+,λ^-) is mutually singular.

Proof: (i) implies (iii): Let (P,N) be a Hahn decomposition for (μ,ν). Then $\lambda^-(P) = 0$, since $\nu \le \mu$ on subsets of P; similarly, $\lambda^+(N) = 0$.

(iii) implies (ii): clear.

(ii) implies (i): Since J is idempotent, (λ^+,λ^-) is its own Jordan decomposition; hence (λ^+,λ^-) is mutually singular, by the argument showing that (i) implies (iii). Let (P,N) split A so that $\lambda^-(P) = 0$, $\lambda^+(N) = 0$. For any $E \subseteq P$ such that $\mu(E) < \infty$, we have

$$\lambda^-(P) \ge \nu(E) - \mu(E).$$

Hence $\nu \le \mu$ on subsets of P. A similar argument yields $\mu \le \nu$ on N. We now have a closed circle of implications, so these three conditions are equivalent. $\square$

We are interested in conditions ~~which~~ That guarantee Hahn decomposability.

Theorem: Let μ, ν be measures. Any of the following three conditions implies that (μ,ν) is Hahn decomposable:

(i) μ is abcont;

(ii) ν is abcont;

(iii) $\inf(\mu,\nu)$ is ~~sigma~~ σ-finite.

Proof: (i) Consider packings of sets $E \in \Sigma$ satisfying $\mu(E) > \nu(E)$. ~~There exists~~ a maximal packing $\mathcal{G}$ of this sort exists ~~— that is~~ i.e., a packing not properly contained in any larger such packing. (This inference requires the axiom of choice, say in the form of Zorn's lemma).

We show that $\mathcal{G}$ must be countable. Since μ is abcont, ~~there exists~~ a finite measure ρ exists with $\mu \ll \rho$. For each $E \in \mathcal{G}$, $\mu(E) > 0$, hence $\rho(E) > 0$. The class of $\mathcal{G}$-sets E on which $\rho(E) > 1/n$ must be finite for each $n = 1, 2, \ldots$, since ρ is finite. Then $\mathcal{G}$ itself, as the union of these classes, must be countable.

We may then write $\mathcal{G} = \{E_1, E_2, \ldots\}$. For each m, $\nu(E_m) < \infty$; hence, restricting everything to E_m, $\mu - \nu$ is an ordinary signed measure, and so has a Hahn decomposition $P_m \cup N_m = E_m$.

We claim that $(P, A\backslash P)$ is a Hahn decomposition for (μ,ν), where

$$P = P_1 \cup P_2 \cup \cdots .$$

To verify this, let $F \subseteq P$; we have

$$\mu(F \cap P_m) \geq \nu(F \cap P_m) \qquad \text{~~(28)~~ (3.1.28)}$$

for each m; by summation, $\mu(F) \geq \nu(F)$; thus $\mu \geq \nu$ on P.

Conversely, let $F \subseteq A\setminus P$; F may be written in the form

$$F = [F\setminus(\cup G)] \cup (F \cap N_1) \cup (F \cap N_2) \cup \cdots . \qquad (3.1.29)\ \cancel{(29)}$$

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We have

$$\nu(F \cap N_m) \geq \mu(F \cap N_m) \qquad (3.1.30)\ \cancel{(30)}$$

for each m. Furthermore, we have

$$\nu(F\setminus\cup G) \geq \mu(F\setminus\cup G) \qquad (3.1.31)\ \cancel{(31)}$$

For, if (31) were false, we could form a larger packing on which $\mu(E) > \nu(E)$ by including the set $F\setminus\cup G$; this contradicts the maximality of G. Adding (31) to the sum of (30), we obtain $\mu(F) \geq \nu(F)$, so that $\nu \geq \mu$ on $A\setminus P$. This concludes the proof.

¶ 18(ii) Same as (i), with rôles of μ, ν interchanged.

¶ 18(iii) Let (A_n), $n = 1, 2, \ldots$, be a partition of universe set A such that

$$\inf(\mu,\nu)(A_n) < \infty,$$

all n. For each n there is then a set $F_n \subseteq A_n$ such that

$$\mu(F_n) < \infty, \quad \nu(A_n\setminus F_n) < \infty. \qquad (3.1.32)\ \cancel{(32)}$$

(32) shows that, when ~~everything is~~ restricted to F_n, or to $A_n\setminus F_n$, $\mu - \nu$ is a signed measure and thus has a Hahn decomposition. Let P be the union of the pieces of the decompositions on which $\mu \geq \nu$, and let N be the union of the pieces on which $\nu \geq \mu$. Then (P,N) is a Hahn decomposition for (μ,ν) by

the argument of ~~(29) above.~~ part (i). |||

24 Exercise: Show that each of the three premises in the preceding theorem implies the following condition, and that this condition in turn implies that (μ,ν) is Hahn decomposable:

There exists a set $E \in \Sigma$ such that μ restricted to E is abcont and ν restricted to $A \setminus E$ is abcont.

18 For all we know to this point, any pair of measures might be Hahn decomposable. The following counterexample scotches this possibility.

Theorem: There exists a pair of measures (μ,ν) ~~which~~ that is not Hahn decomposable, and ~~which~~ that, furthermore, is its own Jordan decomposition.

Proof: Let A be uncountable, and let Σ consist of all countable subsets of A and their complements; split A into two uncountable pieces, P and N (note that P, N are not measurable), and let μ, ν be enumeration measure restricted to P, N, respectively. That is, for $E \in \Sigma$, if $E \cap P$ is finite, then $\mu(E)$ = number of points in $E \cap P$; otherwise, $\mu(E) = \infty$; ν is defined similarly, with N in place of P. One easily checks that these are bona fide measures.

Now suppose $(E, A \setminus E)$ were a Hahn decomposition for (μ,ν). Either E or $A \setminus E$ must be countable. If E is countable, then $P \setminus E$ is non-empty; choosing $a_o \in P \setminus E$, we have

$$\nu\{a_o\} = 0 < 1 = \mu\{a_o\},$$

so $\nu \geq \mu$ is false on $A \setminus E$, contradiction. If $A \setminus E$ is countable, then $N \cap E$ is non-empty; choosing $a_o \in N \cap E$, we have

$$\mu\{a_o\} = 0 < 1 = \nu\{a_o\},$$

so $\mu \geq \nu$ is false on E, contradiction. Hence there is no Hahn decomposition.

18 Next, let $(\lambda^+, \lambda^-) = J(\mu, \nu)$, and let E be a countable set. $\nu(E \cap P) = 0$, hence

$$\lambda^+(E) \geq \mu(E \cap P) - \nu(E \cap P) = \mu(E). \qquad (3.1.33)\ \cancel{(33)}$$

Also, $P \setminus E$ is infinite and hence contains an infinite countable set F. $\nu(F) = 0$, so that

$$\lambda^+(A \setminus E) \geq \mu(F) - \nu(F) = \infty. \qquad (3.1.34)\ \cancel{(34)}$$

Relations (33) and (34) show that $\lambda^+ \geq \mu$, so these measures are equal. A similar argument yields $\lambda^- = \nu$. Thus (μ, ν) is its own Jordan decomposition. ▯

For this counterexample, one easily verifies that $\inf(\mu, \nu)$ takes the value 0 on countable sets, and takes the value ∞ on their complements. Thus $\inf(\lambda^+, \lambda^-)$ is not always 0.

9.9.8

Theorem: Let μ, ν be measures. If the pair (μ, ν) is mutually singular, then (μ, ν) is its own Jordan decomposition.

Proof: Let (P, N) be a measurable partition of A such that

$$\nu(P) = \mu(N) = 0.$$

D

Let $(\lambda^+, \lambda^-) = \underline{J}(\mu,\nu)$. For any $\underline{E} \in \Sigma$, we have $\nu(\underline{E} \cap \underline{P}) = 0$, so

$$\lambda^+(E) \geq \mu(E \cap P) - \nu(E \cap P) = \mu(E \cap P) = \mu(E).$$

Hence $\lambda^+ \geq \mu$, so these are equal. A similar argument yields $\lambda^- = \nu$. ~~This concludes the proof.~~ $|||$ $\square$

#

The counterexample above shows that the converse of this theorem does not always hold.

#

24 ¶ Exercises.

1° ~~(i)~~ Show that the following condition is necessary and sufficient for (μ,ν) to be its own Jordan decomposition:

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For any $\underline{E} \in \Sigma$, if (μ,ν) restricted to E is Hahn decomposable, then (μ,ν) restricted to $\underline{E}$ is mutually singular.

2° ~~(ii)~~ Let (μ,ν) be its own Jordan decomposition. Show that $\theta = \underline{\inf}(\mu,\nu)$ can take only the values 0 and ∞. (Hint: Use (21) to deduce that $2\theta \leq \theta$).

This last exercise may be compared with the result: $\underline{\inf}(\mu,\nu) = 0$ iff (μ,ν) is mutually singular. For a proof see ~~(7.7.7)~~ p. 000 ~~page below.~~

A

3.2. Pseudomeasures

From now on all measures will be ~~sigma~~ σ-finite, unless explicitly noted otherwise. All measures are on the same space, $(\underline{A},\Sigma)$. We ~~shall be~~ are concerned with ordered pairs, (μ,ν), of such measures. Among these pairs, the ~~ones that are~~ mutually singular ones will play a key rôle. The following theorem gives some characterizations of these pairs.

Theorem: Let (μ,ν) be a pair of sigma-finite measures, with Jordan decomposition (λ^+,λ^-). If (μ,ν) has any of the following properties, then it has all of them:

(i) (μ,ν) is mutually singular;

(ii) (μ,ν) is its own Jordan decomposition;

(iii) $\inf(\mu,\nu) = 0$;

(iv) $\lambda^+ = \mu$;

(v) $\lambda^- = \nu$;

(vi) $\lambda^+ + \lambda^- = \mu + \nu$;

(vii) $\lambda^+ + \lambda^- = \sup(\mu,\nu)$;

(viii) $\mu + \nu = \sup(\mu,\nu)$.

Proof: (i) implies (ii): already proved.

(ii) implies (iv) and (v): obvious.

(iv) implies (vi):

$$\lambda^+ + \lambda^- = \mu + \lambda^- = \nu + \lambda^+ = \nu + \mu.$$

(The middle equality is from ~~(13) above~~ (1.13) of section 1). (1.13).) - Ed: O.K.?

(v) implies (vi):

$$\lambda^+ + \lambda^- = \lambda^+ + \nu = \lambda^- + \mu = \nu + \mu.$$

(vi) implies (vii) and (viii):

$$\lambda^+ + \lambda^- \le \mu + \lambda^- = \sup(\mu,\nu) \le \mu + \nu. \qquad (3.2.1)$$

(1.13) ~~of section 1~~;

(The equality is from ~~(13)~~ ~~above;~~ the right-hand inequality follows from the definition of sup(μ,ν))). Since the two extreme expressions in (1) are equal, the middle expressions must equal each of them; this yields (vii) and (viii).

18 ¶ (vii) implies (vi): As is easily verified, the following equality holds in general:

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$$\mu + \nu = \sup(\mu,\nu) + \inf(\mu,\nu). \qquad \cancel{(2)}\ (3.2.2)$$

By assumption, the right side of (2) equals

9.9.9

$$\lambda^{+} + \lambda^{-} + \inf(\mu,\nu) = \lambda^{+} + \nu = \sup(\mu,\nu) = \lambda^{+} + \lambda^{-}. \qquad \cancel{(3)}\ (3.2.3)$$

(1.21) ~~of section 1~~,

The left equality in (3) arises from ~~(21)~~ ~~above~~, and the middle equality

(1.13) ~~of section 1~~.

from ~~(13)~~ ~~above~~; (2) and (3) together yield (vi).

18 ¶ (i) implies (iii): A may be split into P, N such that $\mu(N) = 0$, $\nu(P) = 0$; hence

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$$\inf(\mu,\nu)(A) \le \mu(N) + \nu(P) = 0.$$

18 ¶ (iii) implies ~~(viii)~~ (viii): This follows at once from (2).

18 So far we have made no use of the ~~sigma~~ σ-finiteness assumption. We now use it to show that (viii) implies (i). This establishes a closed circle of implications and shows that all eight properties are logically equivalent.

(viii) implies (i): By ~~sigma~~ σ-finiteness, (μ,ν) has a Hahn decomposition (P,N). Let $\mathcal{G}$ be a countable measurable partition of P such that $\mu(G) < \infty$ for each $G \in \mathcal{G}$. Since $\mu \geq \nu$ on every subset of G, we have

$$\mu(G) = \sup(\mu,\nu)(G) = \mu(G) + \nu(G),$$

implying $\nu(G) = 0$. This is true for each such G, so that $\nu(P) = 0$. A similar argument yields $\mu(N) = 0$. Thus (μ,ν) is mutually singular. |||

Now consider the set $\mathbb{M}$ of all ordered pairs of ~~sigma~~ σ-finite measures (μ,ν) on (A,Σ). Two such pairs are said to be equivalent iff they have the same Jordan decomposition. This equivalence relation determines a partition Ψ of $\mathbb{M}$; ~~namely~~ viz., each element ψ of Ψ is the set consisting of all pairs having some particular (λ^+,λ^-) as their common Jordan decomposition.

Definition~~s~~: Each element $\psi \in \Psi$ is called a pseudomeasure. The common Jordan decomposition of all members of ψ is called the Jordan form of pseudomeasure ψ, and will usually be written as (ψ^+,ψ^-); the measures ψ^+, ψ^- are called, respectively, the upper and lower variations of ψ. Ψ itself is the space of pseudomeasures over (A,Σ).

capital psi

Theorem: Let ψ be a pseudomeasure, and let A be split into two measurable sets P, N. Each of the following conditions implies the other two:

18 (i) (P,N) is a Hahn decomposition for every pair of measures (μ,ν) belonging to ψ;

(ii) (P,N) is a Hahn decomposition for at least one pair (μ,ν) belonging to ψ;

(iii) $\psi^-(P) = \psi^+(N) = 0$.

Proof: (i) implies (ii): Obvious, since ψ is not empty.

18 (ii) implies (iii): $\mu \geq \nu$ on subsets of P, hence $\psi^-(P) = 0$; $\nu \geq \mu$ on subsets of N, hence $\psi^+(N) = 0$.

18 (iii) implies (i): Suppose (i) is false, so that there is a pair $(\mu,\nu) \in \psi$ and a set E such that, say, $E \subseteq P$ and $\mu(E) < \nu(E)$. But then

$$\psi^-(P) \geq \nu(E) - \mu(E) > 0,$$

so that (iii) is false. If, instead, $E \subseteq N$ and $\nu(E) < \mu(E)$, then

$$\psi^+(N) \geq \mu(E) - \nu(E) > 0,$$

so that (iii) is again false.

18 This establishes a closed circle of implications, so the three conditions are logically equivalent. ///

Definition: (P,N) is a Hahn decomposition for pseudomeasure ψ iff any (hence all) of the conditions above are satisfied.

Every pseudomeasure has a Hahn decomposition, since any pair of ~~sigma~~ σ-finite measures is Hahn decomposable.

The basic relations between pseudomeasures and their Jordan forms are spelled out in the following results.

Theorem: The mapping ~~which~~ That associates each pseudomeasure with its Jordan form establishes a 1-1 correspondence between Ψ and the set of mutually singular sigma-finite pairs (μ,ν).

The Jordan form of ψ belongs to ψ. In fact, ψ consists of all pairs of measures of the form $(\psi^+ + \theta,\ \psi^- + \theta)$, where θ ranges over the set of sigma-finite measures.

Proof: ~~Obviously,~~ different pseudomeasures have different Jordan forms, and vice versa, so we ~~have to~~ must show that the set of Jordan forms (ψ^+,ψ^-) coincides with the set of mutually singular measures. If (μ,ν) is a Jordan form, ~~then,~~ by ~~the~~ a preceding theorem, (μ,ν) is mutually singular. Conversely, if (μ,ν) is mutually singular, ~~then~~ it is its own Jordan decomposition. Hence it is the Jordan form of the pseudomeasure to which it itself belongs.

Let (μ,ν) belong to pseudomeasure ψ, so that $(\psi^+,\psi^-) = \underline{J}(\mu,\nu)$. But then, by ~~(21)~~ (1.21) ~~of section 1,~~ ~~above,~~ we have

$$\mu = \psi^+ + \underline{\inf}(\mu,\nu),\quad \nu = \psi^- + \underline{\inf}(\mu,\nu),$$

so that (μ,ν) is indeed of the form $(\psi^+ + \theta,\ \psi^- + \theta)$. Conversely, let (μ,ν) be of this form, and let $(\lambda^+,\lambda^-) = \underline{J}(\mu,\nu)$. Choose $\underline{E} \in \Sigma$, and let $\underline{F} \subseteq \underline{E}$ satisfy: $\psi^-(\underline{F}) + \theta(\underline{F}) < \infty$; then

$$\psi^+(\underline{E}) \geq \psi^+(\underline{F}) \geq [\psi^+(\underline{F}) + \theta(\underline{F})] - [\psi^-(\underline{F}) + \theta(\underline{F})].$$

Taking the supremum over such sets F, we obtain

$$\psi^+(E) \geq \lambda^+(E).$$

To prove the reverse inequality, let (P,N) split A so that $\psi^-(P) = \psi^+(N) = 0$, and let $\mathcal{G}$ be a countable partition of A such that $\theta(G) < \infty$, all $G \in \mathcal{G}$. For any such G we have

$$\psi^-(E \cap G \cap P) = 0 \quad \text{and} \quad \theta(E \cap G \cap P) < \infty,$$

so that

$$\begin{aligned} \lambda^+(E \cap G) &\geq [\psi^+(E \cap G \cap P) + \theta(E \cap G \cap P)] \\ &\quad - [\psi^-(E \cap G \cap P) + \theta(E \cap G \cap P)] \\ &= \psi^+(E \cap G \cap P) = \psi^+(E \cap G). \end{aligned}$$

Adding these inequalities over all $G \in \mathcal{G}$, we obtain

$$\lambda^+(E) \geq \psi^+(E).$$

Thus $\lambda^+ = \psi^+$. A similar argument establishes $\lambda^- = \psi^-$. It follows that (μ,ν) belongs to ψ. ■

Thus a pseudomeasure is a collection of pairs of measures, among which is one special "canonical" pair, the Jordan form. This is the unique pair ~~which~~ that is mutually singular, ~~which~~ is its own decomposition, ~~which~~ and has the smallest left component among all the pairs, ~~and also~~ as well as the smallest right component. (Proof: Let (μ,ν) *not* be the Jordan form, hence not mutually

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singular, hence $\lambda^+ \neq \mu$, $\lambda^- \neq \nu$). The Jordan form may be recovered from any pair (μ,ν) by subtracting out their "common part" $\underline{\inf}(\mu,\nu)$ from each of them.

The following result establishes a very useful criterion for equivalence.

~~¶~~ # **Theorem:** (equivalence theorem) Let μ_1, ν_1, μ_2, and ν_2 be ~~sigma-finite~~ σ-finite measures. The pair (μ_1,ν_1) is equivalent to (μ_2,ν_2) iff

$$\mu_1 + \nu_2 = \nu_1 + \mu_2 . \tag{3.2.4}$$

~~¶~~ # **Proof:** Half of this theorem has already been proved: (4) is implied by $\underline{J}(\mu_1,\nu_1) = \underline{J}(\mu_2,\nu_2)$. See (1.17) & (1.18) ~~((17)-(18) above)~~ of section 1.

Conversely, let (4) ~~(38)~~ hold. Let $\mathcal{G}_i$ be a countable partition of $\underline{A}$ such that $\nu_i(\underline{G}_i) < \infty$ for all $\underline{G}_i \in \mathcal{G}_i$, $\underline{i} = 1, 2$. Letting $\underline{G}_1$, $\underline{G}_2$ be sets from these respective partitions, ~~we~~ note that ν_1 and ν_2 are both finite on subsets of $\underline{G}_1 \cap \underline{G}_2$, hence may be subtracted from both sides of (4) ~~(38)~~ on such sets. This justifies the middle equality in the following chain. Let $(\lambda_i^+, \lambda_i^-) = \underline{J}(\mu_i,\nu_i)$, $\underline{i} = 1, 2$, and choose $\underline{E} \in \Sigma$. Then

$$\begin{aligned}
&\lambda_1^+(\underline{E} \cap \underline{G}_1 \cap \underline{G}_2) \\
&\qquad = \underline{\sup}\{\mu_1(\underline{F}) - \nu_1(\underline{F}) \mid \underline{F} \subseteq (\underline{E} \cap \underline{G}_1 \cap \underline{G}_2)\} \\
&\qquad = \underline{\sup}\{\mu_2(\underline{F}) - \nu_2(\underline{F}) \mid \underline{F} \subseteq (\underline{E} \cap \underline{G}_1 \cap \underline{G}_2)\} \\
&\qquad = \lambda_2^+(\underline{E} \cap \underline{G}_1 \cap \underline{G}_2) .
\end{aligned}$$

Adding over all $\underline{G}_1 \in \mathcal{G}_1$, all $\underline{G}_2 \in \mathcal{G}_2$, we obtain $\lambda_1^{+}(\underline{E}) = \lambda_2^{+}(\underline{E})$. A similar argument yields $\lambda_1^{-} = \lambda_2^{-}$. Hence (μ_1,ν_1) and (μ_2,ν_2) are equivalent. |||

Exercise: Show that this result remains true if the sigma-finiteness assumption is weakened to: $\underline{\inf}(\mu_{\underline{i}},\nu_{\underline{i}})$ is sigma-finite for $\underline{i}$ = 1, 2.

We now make a few notational conventions. Pairs (μ,ν) will generally be used to denote the pseudomeasures to which they belong. Equivalence between pairs will be denoted by the equality sign. Thus $(\mu_1,\nu_1) = (\mu_2,\nu_2)$ does not mean that $\mu_1 = \mu_2$, $\nu_1 = \nu_2$; it means that these pairs belong to the same pseudomeasure, so that only (4) is true. Similarly, we write $(\mu,\nu) = \psi$ to indicate that (μ,ν) belongs to pseudomeasure ψ.

Sigma-finite measures and signed measures may now be thought of as special kinds of pseudomeasures. Specifically, the measure μ may be identified with the pseudomeasure $(\mu,0)$. (Here 0 is the identically zero measure). If μ is a sigma-finite signed measure, let (μ^{+},μ^{-}) be its ordinary Jordan decomposition. We now identify μ with the pseudomeasure (μ^{+},μ^{-}).

Pseudomeasure ψ is bounded iff both ψ^{+} and ψ^{-} are bounded measures. The class of bounded pseudomeasures may be identified with the class of bounded signed measures. Next consider the case where exactly one of ψ^{+} and ψ^{-} is infinite. The class of these pseudomeasures may be identified with the class of infinite

(~~sigma~~ σ-finite) signed measures. Finally, we have the case where both ψ^+ and ψ^- are infinite measures. These "proper" pseudomeasures are new kinds of entities, and provide the rationale for this whole development.

B

The Algebra of Pseudomeasures

We now define various algebraic operations on pseudo-measures. The result we are aiming at is that, under various natural definitions, the set of all pseudomeasures, Ψ, becomes a (real) vector space. First we define addition.

D # **Definition:** The sum of the two pseudomeasures (μ_1, ν_1) and (μ_2, ν_2) is the pseudomeasure $(\mu_1 + \mu_2,\ \nu_1 + \nu_2)$.

This definition is not quite as straightforward as it appears, because the pairs (μ, ν) stand not for themselves but for the pseudomeasures to which they belong. For this definition to be consistent, the pseudomeasure represented by the sum must not depend on the particular pairs chosen for the summands. i.e. ~~That is~~, if another two pairs, (μ_1', ν_1') and (μ_2', ν_2') are respectively equivalent to (μ_1, ν_1) and (μ_2, ν_2), then $(\mu_1' + \mu_2',\ \nu_1' + \nu_2')$ must be equivalent to $(\mu_1 + \mu_2,\ \nu_1 + \nu_2)$. This fact, is, however, an easy consequence of the equivalence criterion just proved, and is left as an exercise. (Note also that the sum of two ~~sigma~~ σ-finite measures is ~~sigma~~ σ-finite.)

We now want to verify that the properties of vector spaces, insofar as they refer to addition, are satisfied by this definition. It is obvious that $\psi_1 + \psi_2 = \psi_2 + \psi_1$, and that

$\psi_1 + (\psi_2 + \psi_3) = (\psi_1 + \psi_2) + \psi_3$. We now need the concepts of zero and negation.

Definition: The <u>zero</u> <u>pseudomeasure</u> is the one whose Jordan form is $(0,0)$.

This is the pair both of whose members are the identically zero measure. We ~~shall~~ denote this pseudomeasure simply by 0, if no confusion is possible. From the equivalence criterion it is immediate that the ~~sigma~~ σ-finite pair (μ,ν) belongs to this pseudomeasure iff $\mu = \nu$.

Definition: The <u>negation</u> of pseudomeasure (μ,ν) is pseudomeasure (ν,μ).

Once again this definition must be checked for consistency: The negation of an equivalent pair must be equivalent to the negation of the original pair. This follows immediately from the equivalence criterion. Negation will be denoted as usual by a minus sign. <u>Subtraction</u> is defined as follows.

Definition: $\psi_1 - \psi_2 = \psi_1 + (-\psi_2)$.

These definitions again satisfy the conditions for a vector space: $\psi + 0 = \psi$ for any pseudomeasure ψ, and $-\psi$ is the unique additive inverse of ψ: $\psi + (-\psi) = 0$.

9.9.12

Next we define <u>scalar</u> <u>multiplication</u>.

Definition: The *product* of the real number b and the pseudo-measure (μ,ν) is the pseudomeasure $(b\mu, b\nu)$ if $b \geq 0$, and is the pseudomeasure $((-b)\nu, (-b)\mu)$ if $b < 0$.

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Here $b\mu$ is, ~~of course,~~ the measure ~~which~~ that assigns the value $b\cdot\mu(E)$ to the measurable set E. Note that measures are always multiplied by non-negative numbers, so that they remain measures. Again, a proof of consistency is required for this operation, and the proof is trivial. The second part of this definition could have been framed in terms of the first part as follows: If $b < 0$, then $b\psi = (-b)(-\psi)$.

The remaining axioms for a vector space may now be verified routinely. For real numbers b_1, b_2, and pseudomeasures ψ_1, ψ_2, $b_1(\psi_1 + \psi_2) = b_1\psi_1 + b_1\psi_2$; $(b_1 + b_2)\psi_1 = b_1\psi_1 + b_2\psi_1$; $b_1(b_2\psi_1) = (b_1b_2)\psi_1$; $1\cdot\psi_1 = \psi_1$. The only minor complications arise with the second equality, where the various sign combinations for b_1, b_2, and $b_1 + b_2$ must be examined. Details are omitted. ~~To~~ We summarize~~:~~ as follows.

Theorem: Under the foregoing definitions the set of all pseudo-measures, Ψ, is a (real) vector space.

As ~~we~~ discussed in Section 2.6, the set of bounded signed measures is a vector space. This property is lost for the larger set of σ-finite signed measures, because addition sometimes leads to the meaningless expression $\infty - \infty$, and is therefore not well-defined for certain pairs. What we have done, in effect, is embed this set in a still larger set, and to extend the domains of addition and scalar multiplication in such a way that the vector space property is restored.

Note that subtraction of <u>measures</u> is compatible with subtraction of <u>pseudomeasures</u> wherever both operations apply. To see this, let $\mu \geq \nu$ be ~~sigma~~ σ-finite measures. The difference $\mu - \nu$ was defined in the preceding section by the relation

$$(\mu - \nu,\ 0) = \underline{J}(\mu,\nu). \qquad \text{(3.2.5)} \ \cancel{(5)}$$

~~Now,~~ identifying μ and ν with the pseudomeasures $(\mu,0)$, $(\nu,0)$, respectively, and using the new definition of subtraction, we obtain

$$(\mu,0) - (\nu,0) = (\mu,0) + (0,\nu) = (\mu,\nu) = (\mu-\nu,0).$$

This last equality follows from (5), or from the equivalence theorem (4) upon noting that

$$\mu + 0 = (\mu-\nu) + \nu.$$

Thus $\mu - \nu$ in the pseudomeasure sense equals the pseudomeasure $(\mu-\nu,0)$, which may be identified with the measure $\mu - \nu$, subtraction being defined as in (5). Neither subtraction concept may be subsumed under the other, however, since their domains of definition differ.

To illustrate these concepts, consider the case of a <u>finite</u> ~~sigma~~ σ-field Σ. Except for the trivial case $\underline{A} = \emptyset$, Σ is generated by a finite partition of universe set A into non-empty sets, say $\underline{A}_1,\ldots,\underline{A}_n$. We claim that the space of pseudomeasures Ψ is here isomorphic to $\underline{n}$-space, the set of all real $\underline{n}$-tuples, with the usual definitions of addition and scalar multiplication. To see this, note first that a measure on $(\underline{A},\Sigma)$ is completely determined by its values on the partition elements $\underline{A}_1,\ldots,\underline{A}_n$. Thus measures may be "coded" by $\underline{n}$-tuples of non-negative numbers. This establishes a 1-1 correspondence between the set of (~~sigma~~ σ-) finite measures on $(\underline{A},\Sigma)$ and the non-negative orthant of $\underline{n}$-space. Furthermore, this correspondence extends in an obvious way to a 1-1 relation between the set of (~~sigma~~ σ-) finite **signed** measures and all of $\underline{n}$-space. If we identify the finite signed measure λ with the set of all pairs of finite measures (μ,ν) such that $\mu - \nu = \lambda$, ~~one easily~~ we can check~~s~~ that this is precisely the operation of gathering these pairs into pseudo-measure classes. The correspondence

pseudomeasures $\leftrightarrow$ signed measures $\leftrightarrow$ $\underline{n}$-tuples

is then ~~easily~~ verified to be an isomorphism among vector spaces, in the sense that it is preserved under addition and scalar multiplication in the respective systems.

There would be little point in constructing the elaborate machinery of pseudomeasure theory if ~~one~~ we were dealing only with finite ~~sigma~~ σ-fields. The point is, ~~of course~~, that these concepts carry over to arbitrary measurable spaces $(\underline{A},\Sigma)$, yielding results that are far from trivial.

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B

Integration with Pseudomeasures

Just as the concept of addition was extended above with the use of pseudomeasures, so will the concept of integration now be extended. Recall that everything is defined on the fixed measurable space $(\underline{A},\Sigma)$.

D

Definition: Let $\underline{f}$ be a real-valued measurable function. The (indefinite) integral of $\underline{f}$ with respect to the pseudomeasure (μ,ν) is the pseudomeasure

$$\left(\int \underline{f}^{+}\, d\mu + \int \underline{f}^{-}\, d\nu \;,\; \int \underline{f}^{-}\, d\mu + \int \underline{f}^{+}\, d\nu\right). \qquad \text{(3.2.6)}\ \cancel{(6)}$$

Expression (6) is to be understood as follows. First ~~of all~~, $\underline{f}^{+}$ and $\underline{f}^{-}$ are the non-negative functions given by

$$\underline{f}^{+}(\underline{a}) = \max(\underline{f}(\underline{a}),\, 0), \quad \underline{f}^{-}(\underline{a}) = \max(-\underline{f}(\underline{a}),\, 0).$$

Next, the four integrals in (6) are ordinary indefinite integrals. The indefinite integral of a non-negative real-valued function with respect to a σ ~~sigma~~-finite measure is itself a σ ~~sigma~~-finite measure. Hence (6) is a pair of σ ~~sigma~~-finite measures, and as such it represents a pseudomeasure.

A consistency question again arises with respect to this definition. Namely, if a pair (μ',ν') equivalent to (μ,ν) is substituted in (6), will the resulting pair be equivalent to the original (6)? The answer is yes, and the proof is again an easy consequence of the equivalence criterion, together with the elementary integration rule,

$$\int g\, d\lambda_1 + \int g\, d\lambda_2 = \int g\, d(\lambda_1 + \lambda_2) \qquad (3.2.7)\ \cancel{(7)}$$

Details are left as an exercise.

Note that (6) is well-defined for <u>any</u> real-valued measurable function and <u>any</u> pseudomeasure. In particular, it is valid for any (~~sigma~~ σ-finite) signed measure, interpreted as a pseudo-measure. This contrasts with the usual definition, which some-times leads to the meaningless expression ∞ − ∞. From our point of view, what happens is that the operation of indefinite integration sometimes leads out of the class of signed measures into the essentially wider realm of pseudomeasures, just as addition sometimes does. It ~~is easily~~ can be seen that, when the ordinary indefinite integral is well-defined (and the integrating signed measure is ~~sigma~~ σ-finite), it yields a signed measure equivalent to ~~(17) above.~~ (6). Thus our definition does, indeed, extend the ordinary definition.

We ~~shall~~ use the notation

$$\int f\, d(\mu,\nu) \quad \text{or} \quad \int f\, d\psi$$

for integration with respect to a pseudomeasure.

Most of the elementary theorems concerning integrals generalize to pseudomeasure integrals. We ~~shall~~ consider a few of these theorems involving equalities in this section. (Theorems involving inequalities will be discussed later).

Theorem: Let $\underline{f}$ be a real-valued measurable function, and let ψ_1 and ψ_2 be pseudomeasures. Then

$$\int f\, d\psi_1 + \int f\, d\psi_2 = \int f\, d(\psi_1 + \psi_2). \qquad \text{(3.2.8)}\ \cancel{(8)}$$

Proof: Choose arbitrary members (μ_i, ν_i) of ψ_i $(i = 1, 2)$; then $(\mu_1 + \mu_2, \nu_1 + \nu_2)$ belongs to $\psi_1 + \psi_2$. Expanding the two sides of (8) with these according to the rule (6), the left (right) side becomes a pair, each measure of which is the sum of four (two) indefinite integrals. Equality of these pairs is established by applying the rule (7) four times, for the non-negative functions $g = f^+$ or $g = f^-$, combined with the measures $\lambda_i = \mu_i$ $(i = 1, 2)$, or $\lambda_i = \nu_i$ $(i = 1, 2)$. |||

Theorem: Let $\underline{f}$ and $\underline{g}$ be real-valued measurable functions, and ψ a pseudomeasure. Then

$$\int f\, d\psi + \int g\, d\psi = \int (f + g)\, d\psi. \qquad \text{(3.2.9)}\ \cancel{(9)}$$

Proof: A rule of the same form as (9) holds for ordinary indefinite integrals with two non-negative functions and a measure. Let (μ, ν) be an arbitrary member of ψ, and expand both sides of (9) by the rule (6). The left side of (9) becomes

gal 10-2

$$\left[\int (f^+ + g^+)\,d\mu + \int (f^- + g^-)\,d\nu,\ \int (f^- + g^-)\,d\mu + \int (f^+ + g^+)\,d\nu\right],$$

while the right side becomes a similar pair with $(f + g)^+$ in place of $(f^+ + g^+)$, and with $(f + g)^-$ in place of $(f^- + g^-)$. Testing by the equivalence criterion (4), we find that these two pairs are equivalent if the following equation holds.

$$f^+ + g^+ + (f + g)^- = f^- + g^- + (f + g)^+. \qquad \cancel{(10)}\ (3.2.10)$$

But the validity of (10) follows at once from the fact that

$$(f^+ - f^-) + (g^+ - g^-) = f + g = (f + g)^+ - (f + g)^-.$$

Theorem: Let f be a real-valued measurable function, ψ a pseudomeasure, and b, c real numbers. Then

$$\int (bf)\, d(c\psi) = bc \int f\, d\psi. \qquad \cancel{(11)}\ (3.2.11)$$

Proof: A rule of the same form as (11) holds for ordinary integrals, and, choosing an arbitrary member (μ,ν) of ψ, expansion of both sides of (11) by rule (6) yields a routine verification. (The four possible sign combinations of b, c must be dealt with separately.)

In explanation of the following theorem, note that since $\int g\, d\psi$ is a pseudomeasure, it makes perfectly good sense to integrate another function f with respect to *it*. The left side of (12) represents the resulting iterated integral, and (12) states that this can actually be expressed by a single integral.

Theorem: (iterated integral theorem) Let f and g be real-valued measurable functions, and ψ a pseudomeasure. Then

$$\int f\,d\left[\int g\,d\psi\right] = \int fg\,d\psi. \qquad (3.2.12)\ \cancel{(12)}$$

Proof: A rule of the same form holds for ordinary integrals for two non-negative functions and a measure. Choosing an arbitrary pair (μ,ν) belonging to ψ, we first expand $\int g\,d\psi$ by (6), and then expand the integral of f by <u>this</u> pair, again by (6). The result is a pair, the left measure of which is

$$\int f^+g^+\,d\mu + \int f^+g^-\,d\nu + \int f^-g^+\,d\nu + \int f^-g^-\,d\mu, \qquad (3.2.13)\ \cancel{(13)}$$

and the right measure of which is obtained from (13) by switching μ and ν. The equality of this pair with the expansion of $\int fg\,d(\mu,\nu)$ follows from the fact that

$$(fg)^+ = f^+g^+ + f^-g^-, \quad \text{and} \quad (fg)^- = f^+g^- + f^-g^+. \qquad (3.2.14)\ (14)$$

(The validity of (14) is established by considering the four possible sign combinations of f, g separately).

These four theorems all follow the same pattern. The equalities (8), (9), (11), (12) are already known to hold for ordinary integrals with non-negative functions and measures, and this fact is used to show that the expansions of the corresponding pseudomeasure integrals are equivalent.

Let $\underset{\sim}{1}$ be the function everywhere equal to 1. The following is easily verified:

$$\int \underset{\sim}{1}\, d\psi = \psi. \qquad \text{(3.2.15)} \quad \cancel{(15)}$$

These results concerning integrals – (8), (9), (12), and (15) – may be summarized in algebraic terms.

Let F be the set of all real-valued measurable functions on $(\underline{A},\Sigma)$. set F is a ring in the algebraic sense, under pointwise addition and multiplication. In fact it is a commutative ring with unity, the unit element being $\underset{\sim}{1}$. Define addition on the space of pseudomeasures Ψ as above; define "scalar multiplication", $\circ$, as a mapping from $F \times \Psi$ to Ψ, viz. ~~namely,~~

$$\underline{f} \circ \psi = \int \underline{f}\, d\psi.$$

centered degree

Then these results, ~~together~~ along with the preceding ones, state that Ψ is a (unitary) module over the ring F with respect to these operations.[3] then Ψ as a real vector space may ~~then~~ be thought of as a module over the subring of constant functions if we identify the real number $\underline{c}$ with the constent function $\underline{f} = \underline{c}$.

We ~~will~~ sometimes need the Jordan form of an integral. This is easily found.

Theorem: Let (ψ^+,ψ^-) be the Jordan form of pseudomeasure ψ, and let $\underline{f}: \underline{A} \to$ reals be measurable. Then

$$\left[\int \underline{f}^+\, d\psi^+ + \int \underline{f}^-\, d\psi^-,\ \int \underline{f}^-\, d\psi^+ + \int \underline{f}^+\, d\psi^-\right] \qquad \text{(3.2.16)} \quad \cancel{(16)}$$

is the Jordan form of $\int_A f\, d\psi$.

Proof: It is clear that the pair (16) belongs to pseudomeasure $\int_A f\, d\psi$. The only thing left to prove is that (16) is mutually singular. Split A into P, N so that $\psi^-(P) = \psi^+(N) = 0$. Then the two left integrals in (16) are zero on the set

$$\left(N \cap \{a \mid f(a) \geq 0\}\right) \cup \left(P \cap \{a \mid f(a) < 0\}\right),$$

while the two right integrals ~~in (16)~~ are zero on the complementary set

$$\left(P \cap \{a \mid f(a) \geq 0\}\right) \cup \left(N \cap \{a \mid f(a) < 0\}\right).$$

~~This completes the proof.~~

Definition: The total variation of pseudomeasure ψ is the measure $\psi^+ + \psi^-$.

This is a direct generalization of the same concept for signed measures. We shall denote the total variation by $|\psi|$.

Next, recall that, if μ and ν are two measures, μ is said to be absolutely continuous with respect to ν iff, for any measurable set E, if $\nu(E) = 0$, then $\mu(E) = 0$. The notation for this is: $\mu << \nu$. We now extend this concept to pseudomeasures.

Definition: Let ψ_1 and ψ_2 be pseudomeasures. ψ_1 is absolutely continuous with respect to ψ_2 iff $|\psi_1| << |\psi_2|$.

This is well-defined and not circular, because $|\psi_1|$ and $|\psi_2|$ are ordinary measures. The same notation will be used:

$\psi_1 << \psi_2$.

We end this subsection with two generalizations of well-known theorems.

Theorem: Let f be a real-valued measurable function, and ψ a pseudomeasure. Then

$$\left|\int f\, d\psi\right| = \int |f|\, d|\psi| .$$

(Here $|f| = \max(f, -f)$, the absolute value of f. The expression on the right is an ordinary indefinite integral, and the claim is that it equals the total variation of the pseudomeasure $\int f\, d\psi$).

Proof: Since (16) is the Jordan form of $\int f\, d\psi$, the total variation $|\int f\, d\psi|$ is the sum of the four integrals in (16), which is

$$\int (f^+ + f^-)\, d(\psi^+ + \psi^-) = \int |f|\, d|\psi| . \quad \square$$

Theorem: (Radon-Nikodym theorem for pseudomeasures) Let ψ_1 and ψ_2 be pseudomeasures. ~~There~~ exists a real-valued measurable function f such that $\psi_1 = \int f\, d\psi_2$ iff $\psi_1 << \psi_2$.

Proof: The "only if" part is simple. Let $\psi_1 = \int f\, d\psi_2$, and let E be a measurable set such that $|\psi_2|(E) = 0$. Hence $\psi_2^+(E) = 0$ and $\psi_2^-(E) = 0$. We have proved ~~above~~ that (16) is the Jordan form for an integral $\int f\, d\psi$. Since both measures in (16) clearly equal zero at E for $\psi = \psi_2$, we have $\psi_1^+(E) = \psi_1^-(E) = 0$, so that $|\psi_1|(E) = 0$. This proves that $\psi_1 << \psi_2$.

Conversely, assume $\psi_1 << \psi_2$, so that $|\psi_1| << |\psi_2|$. These are both ~~sigma~~ σ-finite measures, hence, by the ordinary Radon-Nikodym theorem, ~~there~~ exists a non-negative real-valued measurable function g such that $|\psi_1| = \int g \, d|\psi_2|$.

Now let (P_i, N_i) be a Hahn decomposition for ψ_i $(i = 1, 2)$, and define the function f as follows:

$$f(a) = g(a) \quad \text{if } a \in (P_1 \cap P_2) \cup (N_1 \cap N_2),$$

$$f(a) = -g(a) \quad \text{if } a \in (P_1 \cap N_2) \cup (N_1 \cap P_2).$$

We ~~will~~ now show that f is the required function: $\psi_1 = \int f \, d\psi_2$. Expanding the integral in the form (16) for $\psi = \psi_2$, we ~~will~~ show that the pair of measures in (16) is, in fact, (ψ_1^+, ψ_1^-). It suffices to prove this equality for measurable subsets of each of the four sets $(P_1 \cap P_2)$, $(N_1 \cap N_2)$, $(P_1 \cap N_2)$, and $(N_1 \cap P_2)$, ~~for~~ because, since these partition A, equality for any measurable set follows by summation. We will carry out the analysis for $P_1 \cap N_2$, the argument for the other three sets being similar. Since $\psi_1^-(P_1) = 0$ and $\psi_2^+(N_2) = 0$, it follows that $|\psi_1| = \psi_1^+$ and $|\psi_2| = \psi_2^-$ when all measures are restricted to $P_1 \cap N_2$. Also, ~~from (32),~~ f is non-positive on $P_1 \cap N_2$, so that $f^+ = 0$ and $f^- = g$ on this set. Hence, restricted to $P_1 \cap N_2$, the four integrals in ~~(30)~~ (16) (with $\psi = \psi_2$) reduce to $(0 + \int g \, d\psi_2^-, 0 + 0)$. Now $\psi_1^- = 0$ on $P_1 \cap N_2$, while $\psi_1^+ = |\psi_1| = \int g \, d|\psi_2| = \int g \, d\psi_2^-$ on $P_1 \cap N_2$, proving equality. $\square$

(B)

Applications of Pseudomeasures

Having formally introduced pseudomeasures, ~~let us~~ we consider some of the ways they can be used. A number of the following examples have already been mentioned, but we can now ~~give a~~ discuss them more coherent ly ~~discussion of them~~. We ~~shall~~ assume that all measures discussed are ~~sigma~~ σ-finite.

Let μ and ν be measures over Space~~, S,~~ as universe set, with the interpretation: that $\mu(E) \equiv$ gross production of a certain resource in region E, and $\nu(E)$ = gross consumption of that resource in E. If both measures are infinite, they cannot be subtracted to yield net production. We can, however, represent net production by the pseudomeasure (μ,ν). What can be done with this representation? equals

Consider first ~~of all~~ the Jordan form (λ^+,λ^-) of this pseudomeasure, with a Hahn decomposition (P,N). We know that $\mu \geq \nu$ when both are restricted to P, and that $\nu \geq \mu$ when both are restricted to N. Thus (P,N) splits Space into the region of net production and the region of net consumption, and λ^+, λ^- give these respective "net" measures. When these pseudomeasures reduce to ordinary measures or signed measures (which occurs when λ^+ and λ^- are not both infinite), they do so in an intuitively appealing way. For example, suppose production is everywhere 3 times consumption. The pseudomeasure $(3\nu,\nu)$ has the Jordan form $(2\nu,0)$, which is the ordinary measure 2ν, and states that net production is 2 times consumption.

Problems involving infinite "gross" measures often arise when the horizon is unlimited: the infinite plane of location theory with unlimited space horizon, or economic development programs with unlimited time horizon. In such situations it is convenient (though not usually essential) to frame the problem in the form: "find the optimal pseudomeasure such that ..."

We have just discussed one broad category of application for pseudomeasures: the representation of physical situations. Another, perhaps more important, application is to the representation of preferences. Consider, e.g., ~~for example,~~ an economic development program with infinite horizon. Typically, one represents the "payoff" from a policy p by an integral of the form

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$$\int_0^\infty f(p,t)\,dt, \qquad \text{(3.2.17)} \; (17)$$

where $f(p,t)$, for example, may be determined by total consumption under policy p at time t. One chooses the attainable policy that ~~which~~ maximizes (17). There are two difficulties with an objective function of the form (17). First, suppose the value $+\infty$ can be attained with several policies. Are these to be considered equally good? Simple examples suggest otherwise. Suppose ~~that~~ policies p' and p'' are such that $f(p',t) > f(p'',t)$ for all t, but ~~that~~ both policies give the value $+\infty$ in (17). Intuitively, we ~~one~~ would be inclined to say that p' is the better policy. This means that (17) does not properly represent the structure of preferences. ~~at its upper limit.~~

The second difficulty appears to be even more troublesome. What about feasible policies for which (17) is not well-defined ~~— that is,~~ i.e., where its evaluation leads to the meaningless expression $\infty - \infty$. These policies would simply be incomparable with others under the objective function (17). Yet in many cases simple intuition does suggest that some of these policies are better tha~~t~~n others ~~— for example~~ e.g., when they are related as p' and p" above. Thus again (17) does not properly represent the structure of preferences.

Function (17) is an integral over ~~T~~time. But the same problems can arise with integrals over Space, ~~or~~ Space-Time, or abstract spaces.

Are these difficulties serious? One can of course frame models ~~which~~ that avoid them, and insure that all integrals (17) ~~which~~ that arise are well-defined and finite. (This is done in practice by tr~~a~~uncating at a finite horizon, introducing time-discounts, etc.) But these restrictions prevent one from coming to grips with many significant problems. Several of these arise in location theory and will be taken up later ~~in this book~~. We ~~shall~~ mention one or two others here.

Consider the problem of global welfare maximization. We adopt a terminology and point of view ~~which~~ that is currently out of fashion. Suppose ~~one wants~~ we wish to maximize the balance of total "pleasure" over total "pain" in the world. Both ~~of~~ the foregoing difficulties may arise. Because the time horizon is infinite, all integrals may diverge to $+\infty$ no matter what policy is followed. (A pessimist would maintain that all integrals diverge to $-\infty$, which creates the same difficulty). And, if the total amount of both "pleasure" and "pain" is infinite, none of the integrals will be well-defined.

A rather different example is that of Bernoullian (or von Neumann-Morgenstern) utility.[4] Abstractly, one is given a measurable space, (A,Σ), and the problem is to characterize these preference orderings that a "rational" man might entertain over the set of all possible probability measures on (A,Σ). The main result is the "expected utility" principle: for a rational man there is a measurable function $u: A \to$ reals, such that he prefers probability μ_1 over μ_2 iff

$$\int_A u\, d\mu_1 > \int_A u\, d\mu_2. \qquad (3.2.18)\ ~~(18)~~$$

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Now there is no difficulty if u is a bounded function, for then the integrals in (18) are always finite. If u is, ~~say,~~ unbounded above, however, we ~~one~~ can show there are probability measures for which the integrals (18) $= +\infty$. And if u is unbounded both above and below, there are probabilities for which the integrals (18) are not well defined.

~~Now,~~ We shall argue below that there is no compelling reason why u *should* be bounded. There are perfectly reasonable preference orderings ~~which~~ that call for an unbounded utility function u. But in this case what are we to make of the integrals in (18), and how are we to compare them? One possibility is to restrict comparisons to probability measures ~~which are~~ concentrated on a finite number of points, for with these the integrals (18) are finite even if u is unbounded. This unduly restrictive solution may be avoided, however, if we interpret these integrals as pseudomeasures, and the order relation as standard ordering of pseudomeasures. All this will be fully explained below, and an axiomatic justification for this procedure will be given for the case of a countable universe set.

We ~~shall~~ also discuss the ideas of Ramsey, and of succeeding writers such as Weizsäcker and Gale, on how to deal with unbounded sums and integrals. ~~It turns out that~~ these ideas — the so-called "overtaking" criteria — drop out as special cases of the development below. Thus, the use of pseudomeasure-valued utility indicators leads to a unified theory that ~~which~~ includes not only one-dimensional unbounded integrals (the "overtaking" case) but also higher-dimensional cases (such as spatial integrals in location-theory), and, at the same time, incorporates Bernoullian utility theory with unbounded utility functions u.

Starting from ordinary integrals like ~~of the sort~~ (17) or (18), the first step is to go from the definite to the indefinite integral. In comparing unbounded integrals, merely taking note of the value $+\infty$ loses essential information. We should ~~One wants to~~ take into account the entire distribution pattern.

The second step is to note that an indefinite integral is a signed measure. The fact that it appears in the form of an indefinite integral is irrelevant for the following analys~~e~~is. The problem has become one of comparing signed measures.

The third step is to allow for integrals ~~which~~ that are not well-defined in the ordinary sense by interpreting them as pseudomeasures. We recall that $\int \underline{f}\, d\mu$ is always a well-defined pseudomeasure for any real-valued measurable $\underline{f}$ and any pseudomeasure μ. (In the examples above, μ is just an ordinary measure — Lebesgue measure for the development problem, and a probability measure for the Bernoulli problem). Thus we allow pseudomeasure-valued objective functions.

We now have a problem ~~which~~ that embraces all the others as special cases: Develop a plausible criterion for deciding when one pseudomeasure is larger — or "better" — than another. Our investigation will be guided partly by intuition, and also by the requirement that, when the pseudomeasures reduce to bounded signed measures, the ordering ~~of them~~ should be compatible with that induced by the comparison of finite definite integrals.

90 10-4 We have introduced pseudomeasures in connection with the difficulty of ill-defined integrals. ~~It turns out,~~ however, ~~that~~ pseudomeasures are also essentially involved in the difficulty of comparing unbounded integrals. From our point of view, both these difficulties are the same, and, insofar as our program is successful, both are resolved in one stroke.

3.3. ~~The Comparison~~ of Pseudomeasures: Narrow and Standard Ordering

The problem we have just formulated is: Given pseudo-measures ψ_1 and ψ_2, to give a rule for deciding when ψ_1 is to be considered larger than ψ_2. Our discussion will proceed through stages of increasing concreteness. First we take up comparisons in general, with a discussion of partial orderings. Our development makes essential use of the fact that the pseudo-measures are a vector space, and, accordingly, we next discuss partial orderings on vector spaces. We then come to the space of pseudo-measures itself, and the discussion goes through several more stages.

Partial Orderings in General[5]

Let H be a set. A relation on H is a subset of the cartesian product $H \times H$. The particular kinds of relations we are interested in are called partial orders, and will be denoted by $\geq$ or $\succeq$. If x and y are members of H, the notation $x \geq y$ will indicate that the ordered pair (x, y) is in the relation $\geq$. $y \leq x$ means the same thing.

Definition: The relation $\geq$ is a partial order iff

(i) for all $x, y, z \in H$, if $x \geq y$ and $y \geq z$, then $x \geq z$ (transitivity); ~~and~~

(ii) for all $x \in H$, $x \geq x$ (reflexivity).

The interpretations we have in mind for $\geq$ refer to "size" or to "preferredness", and the statement $x \geq y$ may then be read: "x is at least as big as – or, at least as good as – y", depending on context. Transitivity and reflexivity are conditions with obvious intuitive appeal under such interpretations.

Given the partial order $\geq$ on H, we now define two further relations. Let $x, y \in H$.

Definitions: $x > y$ iff $x \geq y$, but not $y \geq x$ (strict order). $x \sim y$ iff $x \geq y$ and $y \geq x$ (indifference).

relation $x > y$ may be read: "x is greater than – or, better than – y". $x \sim y$ may be read: "x is as big as – or, indifferent to – y".

For any pair of elements x, y there are now four possibilities, exactly one of which must hold: (i) $x \sim y$; (ii) $x > y$; (iii) $x < y$ (~~that is~~ i.e., $y > x$); (iv) none of these, which occurs when $x \geq y$ and $y \geq x$ are both false. In the first three cases we say that x and y are comparable (under the relation $\geq$), in the last case, incomparable.

Definitions: A partial order $\geq$ on H is complete iff any pair of elements of H are comparable. $\geq$ is anti-symmetric iff $x \sim y$ implies $x = y$, for all $x, y \in H$.

Thus a partial order is complete iff for any pair $x, y \in H$ either (x,y) or (y,x) (or both) stand in the relation $\geq$. A partial order is anti-symmetric iff no two distinct elements are indifferent. ($x \sim x$ is always true, owing to reflexivity).

Here are some examples. The natural ordering on the real numbers is both complete and anti-symmetric. In utility theory one customarily assumes that a decision-maker's preference ordering is complete, but not necessarily anti-symmetric. Suppose each of a set I of different people has a preference ordering $\succcurlyeq_i$ over a set of alternatives H ($i \in I$). The **Pareto ordering**, $\succcurlyeq$, determined by these is given by: $x \succcurlyeq y$ iff $x \succcurlyeq_i y$ for all $i \in I$. This need not be either complete or anti-symmetric. In what follows we ~~shall~~ make no assumptions concerning completeness or anti-symmetry.

Definition: Given partial order $\geq$ on H, a point $x^o \in H$ is **greatest**, or **best**, iff $x^o \geq x$ for all $x \in H$. Point $x^o \in H$ is **unsurpassed**[6] iff there is no point $x \in H$ such that $x > x^o$.

The following result is immediate.

Theorem: In partial order $\geq$ over H, x^o is greatest iff x^o is unsurpassed and comparable to all other $x \in H$.

Thus any greatest element is unsurpassed, and, if $\geq$ is complete, the two concepts coincide. There may not be a greatest, or even an unsurpassed, element; or ~~on the other hand,~~ there may be several. Any two greatest elements must be indifferent; any two unsurpassed elements must be either indifferent or incomparable.[7]

Definition: Let $\geq_1$ and $\geq_2$ be two partial orderings on set H. $\geq_2$ **extends** $\geq_1$ iff, for all x, $y \in H$,

(i) $x >_1 y$ implies $x >_2 y$, ~~and~~

(ii) $x \sim_1 y$ implies $x \sim_2 y$.

That is, $\geq_2$ extends $\geq_1$ iff, whenever two elements are comparable under $\geq_1$, that order relation is retained under $\geq_2$. In our ensuing discussion we ~~shall~~ place a number of partial orders on the space of all pseudomeasures, each an extension of the preceding.

Definition: Let $f: H_1 \to H_2$ be a function, and let $\geq_2$ be a partial order on H_2. The partial order induced on H_1 by f is the relation $\geq_1$ on H_1 satisfying: $x \geq_1 y$ iff $f(x) \geq_2 f(y)$, for all x, $y \in H_1$.

One easily verifies that $\geq_1$ is, indeed, a partial order. Note that the induction here is backwards, from the range space to the domain.

Definition: Let $(H_1, \geq_1)$ and $(H_2, \geq_2)$ be two partially-ordered spaces. $(H_1, \geq_1)$ is representable by $(H_2, \geq_2)$ iff there is a function $f: H_1 \to H_2$ such that $x \geq_1 y$ iff $f(x) \geq_2 f(y)$, for all x, $y \in H_1$.

These two definitions underlie the representation of preferences by utility functions, for example. Here the space H_2 is usually the real numbers, and $\geq_2$ their natural ordering.

H_1 is the space of possible alternatives for the problem in hand, and f ~~is~~ the utility function. In our discussion H_2 will be the space of pseudomeasures, Ψ, and $\geq_2$ will be one or another of the partial orders to be specified. Our claim is that this space provides a convenient representation for some problems in which preferences are not conveniently representable — or not representable at all — by the real numbers.

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Partial Orderings on Vector Spaces

Let V be a vector space, so that there is an operation of addition (from $V \times V$ to V) and of scalar multiplication (from the real numbers $\times V$ to V). We are interested in a certain restricted class of partial orders on V.

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Definition: A relation $\geq$ on a vector space V is a vector partial order iff

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18 (i) $\geq$ is a partial order in the ordinary sense; ~~and~~

(ii) if $x \geq y$, then $x + z \geq y + z$, for all $x, y, z \in V$; ~~and~~

(iii) if $x \geq y$, and b is a positive real number, then $bx \geq by$.

As an example, take n-space, with the definition: $(x_1, \ldots, x_n) \geq (y_1, \ldots, y_n)$ iff $x_i \geq y_i$ for all $i = 1, \ldots, n$ (the second $\geq$ referring to the natural ordering of the real numbers).

~~It turns out that~~ vector partial orders may be characterized in a very simple and useful fashion.[8] First, we need one more concept.

Definition: Subset P of a vector space V is a convex cone iff

(i) $0 \in P$; and

(ii) if $x \in P$ and $y \in P$, then $x + y \in P$; and

(iii) if b is a positive real number, and $x \in P$, then $bx \in P$.

Theorem: Relation $\geq$ is a vector partial order on the vector space V iff there is a convex cone P such that, for all x, $y \in V$,

$$x \geq y \text{ iff } x - y \in P. \quad (3.3.1)$$

Letting $y = 0$ in (1), it is clear that $P = \{x | x \geq 0\}$. This is called the positive cone of the ordering $\geq$. If P is an arbitrary convex cone, and we use (1) to define the relation $\geq$, it follows that this relation is a vector partial order. This is~~, in fact,~~ generally the most convenient way to specify vector partial orderings.

One easily verifies that $x > y$ iff $(x - y) > 0$, and that $x \sim y$ iff $(x - y) \sim 0$, for vector partial orders.

Narrow Ordering of Pseudomeasures

We now come to the vector space, Ψ, of all pseudomeasures over a fixed measurable space (A,Σ). Here ~~In this subsection~~ we ~~shall~~ define a vector partial ordering called narrow order; in the next subsection, we define another ~~one~~ called standard order, which extends narrow order; and finally, we define a variety of extended orderings, which all extend standard order.

If μ, ν are a pair of measures, or signed measures, we have already used the notation $\mu \geq \nu$ to abbreviate the condition:

$\mu(\underline{E}) \geq \nu(\underline{E})$, for all measurable sets $\underline{E}$. We now define an ordering on pseudomeasures ~~which~~ that generalizes this relation, and ~~which in fact~~ reduces to it when both pseudomeasures are signed measures.

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Definition: The narrow order, $\geq$, on the space of pseudomeasures, is the vector partial order whose positive cone is $\{\psi | \psi^- = 0\}$.

The pseudomeasures whose lower variation is zero are precisely those which are measures, so that the narrow order is the one whose positive cone is the set of (~~sigma~~ σ-finite) measures. (One verifies immediately that this set is, in fact, a convex cone, so that the definition is consistent).

The following theorem gives several necessary and sufficient conditions for two pseudomeasures to be related by the narrow ordering $\geq$. These conditions are all in the form $\mu \geq \nu$, where μ and ν are measures, and this is to be interpreted in the ordinary sense that $\mu(\underline{E}) \geq \nu(\underline{E})$ for all $\underline{E} \in \Sigma$.

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Theorem: (i) Let (μ,ν) be a pseudomeasure; $(\mu,\nu) \geq 0$ iff $\mu \geq \nu$.

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(ii) Let (μ_1,ν_1) and (μ_2,ν_2) be pseudomeasures; $(\mu_1,\nu_1) \geq (\mu_2,\nu_2)$ iff $\mu_1 + \nu_2 \geq \nu_1 + \mu_2$.

(iii) Let ψ_1, ψ_2 be pseudomeasures; $\psi_1 \geq \psi_2$ iff $\psi_1^+ \geq \psi_2^+$ and $\psi_1^- \leq \psi_2^-$.

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Proof: (i) Let (λ^+,λ^-) be the Jordan form of (μ,ν). $(\mu,\nu) \geq 0$ iff $\lambda^- = 0$. It is immediate from the definition of λ^- that

$\mu \geq \nu$ implies $\lambda^- = 0$. Conversely, if $\nu(E) > \mu(E)$ for some measurable E, then $\lambda^-(E) \geq \nu(E) - \mu(E) > 0$; hence $(\mu,\nu) \not\geq 0$.

(ii) Since $\geq$ is a vector partial order, $(\mu_1,\nu_1) \geq (\mu_2,\nu_2)$ iff $(\mu_1 + \nu_2,\ \nu_1 + \mu_2) = (\mu_1,\nu_1) - (\mu_2,\nu_2) \geq 0$. (The equality comes from the equivalence theorem). The result now follows from ~~part~~ (i).

(iii) If $\psi_1^+ \geq \psi_2^+$ and $\psi_1^- \leq \psi_2^-$, then $\psi_1^+ + \psi_2^- \geq \psi_1^- + \psi_2^+$, so that $\psi_1 \geq \psi_2$ from ~~part~~ (ii).

Conversely, suppose $\psi_1^+ \geq \psi_2^+$ is false, so that $\psi_1^+(E) < \psi_2^+(E)$ for some measurable set E. Let (P_2,N_2) be a Hahn decomposition for ψ_2. Then

$$\psi_1^+(E \cap P_2) + \psi_2^-(E \cap P_2) = \psi_1^+(E \cap P_2) \quad (\text{since } \psi_2^-(P_2) = 0)$$
$$\leq \psi_1^+(E) < \psi_2^+(E) = \psi_2^+(E \cap P_2) \quad (\text{since } \psi_2^+(N_2) = 0)$$
$$\leq \psi_2^+(E \cap P_2) + \psi_1^-(E \cap P_2).$$

Hence $\psi_1^+ + \psi_2^- \not\geq \psi_1^- + \psi_2^+$, so that $\psi_1 \not\geq \psi_2$, from ~~part~~ (ii).

Finally, if $\psi_1^-(E) > \psi_2^-(E)$ for some measurable E, let (P_1,N_1) be a Hahn decomposition for ψ_1. An argument similar to the one just given, but with $E \cap N_1$ in place of $E \cap P_2$, again shows that $\psi_1 \geq \psi_2$ is false. □

It now follows ~~easily~~ that the narrow ordering $\geq$ reduces to the ordinary $\geq$ when the pseudomeasures are ordinary signed measures. For, letting μ and ν be signed measures, and identifying them with the pseudomeasures (μ^+,μ^-), (ν^+,ν^-), respectively, we have that $(\mu^+,\mu^-) \geq (\nu^+,\nu^-)$ iff $\mu^+ \geq \nu^+$ and $\mu^- \leq \nu^-$, which is necessary and sufficient for $\mu \geq \nu$ in the ordinary sense. Our notation is therefore consistent.

Narrow order is anti-symmetric. For if $\psi_1 \geq \psi_2$ and $\psi_2 \geq \psi_1$ are both true, ~~then~~ $\psi_1^+ \geq \psi_2^+ \geq \psi_1^+$, and $\psi_1^- \leq \psi_2^- \leq \psi_1^-$, so that $\psi_1^+ = \psi_2^+$ and $\psi_1^- = \psi_2^-$; i.e. ~~— that is~~, $\psi_1 = \psi_2$.

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It follows that $\psi_1 > \psi_2$ iff $\psi_1 \geq \psi_2$ and $\psi_1 \neq \psi_2$. Applied to the theorem above, this yields criteria for one pseudomeasure being bigger than another. e.g., ~~For example,~~ the pseudomeasure $(\mu,\nu) > 0$ iff $\mu \geq \nu$, and $\mu(E) > \nu(E)$ for at least one measurable set E.[9]

Theorem: Narrow order is incomplete, except when Σ is the trivial σ-field $\{A, \emptyset\}$.

Proof: The case $\Sigma = \{A,\emptyset\}$ is left as an exercise. (The space of pseudomeasures is isomorphic to the real numbers in this case, if $A \neq \emptyset$.)

If Σ is not trivial, there is a measurable E^o such that neither E^o nor $A\backslash E^o$ is empty. Choose points $a \in E^o$, $b \in A\backslash E^o$, and define the measures μ, ν by

$$\mu(F) = 1 \text{ if } a \in F, \ \mu(F) = 0 \text{ otherwise, and}$$
$$\nu(F) = 1 \text{ if } b \in F, \ \nu(F) = 0 \text{ otherwise, for}$$

all $F \in \Sigma$. Then $\mu(E^o) = \nu(A\backslash E^o) = 1$, and $\nu(E^o) = \mu(A\backslash E^o) = 0$, so that μ and ν are not zero, and they are mutually singular. Hence the pseudomeasure (μ,ν) is not comparable to 0, and the narrow order $\geq$ is incomplete. $|||$

With the aid of narrow order, we can generalize the standard inequality theorems for integrals to pseudomeasure integrals. In the following, $\geq$ or $>$ when used between pseudomeasures refers to narrow order, while the expression $f \geq g$ between *point* functions f and g means that $f(a) \geq g(a)$ for all $a \in A$.

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Theorem: (inequalities for pseudomeasure integrals). Let ψ_1 and ψ_2 be pseudomeasures, and let f and g be real-valued measurable functions, all on measurable space (A,Σ).

(i) If $\psi_1 \geq 0$ and $f \geq 0$, then $\int f \, d\psi_1 \geq 0$.

(ii) If $\psi_1 \geq 0$ and $f \geq g$, then $\int f \, d\psi_1 \geq \int g \, d\psi_1$.

(iii) If $\psi_1 \geq \psi_2$ and $f \geq 0$, then $\int f \, d\psi_1 \geq \int f \, d\psi_2$.

(iv) If $\psi_1 > 0$ and $f(a) > 0$ for all $a \in A$, then $\int f \, d\psi_1 > 0$.

(v) If $\psi_1 > 0$ and $f(a) > g(a)$ for all $a \in A$, then

$$\int f \, d\psi_1 > \int g \, d\psi_1.$$

(vi) If $\psi_1 > \psi_2$ and $f(a) > 0$ for all $a \in A$, then

$$\int f \, d\psi_1 > \int f \, d\psi_2.$$

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Proof: (i) By assumption, $\psi_1^- = 0$ and $f^- = 0$, so the expansion of $\int f\, d\psi_1$ by ~~(16) of the preceding section~~ (2.16) reduces to $\left\{\int f^+\, d\psi_1^+,\ 0\right\}$. Hence $\left\{\int f\, d\psi_1\right\}^- = 0$, so that $\int f\, d\psi_1 \geq 0$.

(ii) $\int f\, d\psi_1 - \int g\, d\psi_1 = \int (f-g)\, d\psi_1 \geq 0$, from (i). Hence $\int f\, d\psi_1 \geq \int g\, d\psi_1$.

(iii) Since $\psi_1 \geq \psi_2$, $(\psi_1 - \psi_2) \geq 0$. $\int f\, d\psi_1 - \int f\, d\psi_2 = \int f\, d(\psi_1 - \psi_2) \geq 0$, from (i). Hence $\int f\, d\psi_1 \geq \int f\, d\psi_2$.

(iv) Since $\psi_1 > 0$, $\psi_1^+(A) > 0$; also f is positive, so, by a standard integration theorem, $\int_A f\, d\psi_1^+ > 0$. Hence, $\int f\, d\psi_1 = \left\{\int f^+\, d\psi_1^+,\ 0\right\} > 0$. zero

(v) $\int f\, d\psi_1 - \int g\, d\psi_1 = \int (f-g)\, d\psi_1 > 0$, from (iv). Hence $\int f\, d\psi_1 > \int g\, d\psi_1$.

(vi) Since $\psi_1 > \psi_2$, $(\psi_1 - \psi_2) > 0$. Hence $\int f\, d\psi_1 - \int f\, d\psi_2 = \int f\, d(\psi_1 - \psi_2) > 0$, from (iv), so that $\int f\, d\psi_1 > \int f\, d\psi_2$. |||

(B)

Standard Ordering of Pseudomeasures

While narrow order is quite useful, it is literally too narrow to represent preferability relations among pseudomeasures in an intuitively plausible way. Consider, for example, the representation of preferences by definite integrals over Time:

$$U(p) = \int_0^\infty f(p,t)\,dt, \qquad (3.3.2)\ \cancel{(2)}$$

p being a policy. Going from definite to indefinite integrals, and thence to signed measures, ~~one~~ we can translate the criterion (2) as follows. With each policy p is associated a signed measure, μ_p. Policy p_1 is at least as good as policy p_2 iff $\mu_{p_1}(A) \geq \mu_{p_2}(A)$. Note that the value of μ_p on the universe set A is all that counts — this corresponds precisely to the use of the definite integral in (2). We have argued that this sort of criterion is counterintuitive when both signed measures are infinite (of the same sign). But it is perfectly adequate for comparing policies if their corresponding signed measures are both finite.

~~What~~ We would like, then, ~~is~~ an ordering on the space of pseudomeasures such that, when two pseudomeasures are ~~in fact~~ finite signed measures μ, ν, the relation between these agrees with the criterion above: μ is at least as good as ν iff $\mu(A) \geq \nu(A)$. Narrow order does not accomplish this: Take any two finite non-zero measures ~~which are~~ mutually singular. (These always exist if the ~~sigma~~ σ-field Σ is not trivial). These are not comparable under narrow ordering but ~~they~~ are comparable under the criterion just stated.

Standard order, which we are about to define, meets these desiderata. We first need a preliminary result.

g10-11

Lemma: Given measurable space $(\underline{A}, \Sigma)$, the set of pseudomeasures ψ satisfying

$$\psi^+(\underline{A}) \geq \psi^-(\underline{A}), \text{ and } \psi^-(\underline{A}) < \infty \qquad (3.3.3)\ \cancel{(3)}$$

is a convex cone.

Proof: Clearly 0 belongs to this set. If ψ is a pseudomeasure and $\underline{b}$ a positive real number, then $(\underline{b}\psi)^+ = \underline{b}\cdot\psi^+$, and $(\underline{b}\psi)^- = \underline{b}\cdot\psi^-$. Hence if ψ belongs, then $\underline{b}\psi$ belongs.

Finally, suppose ψ_1, ψ_2 both belong; we must show that $\psi_1 + \psi_2$ belongs. First, from (3), $\psi_{\underline{i}}^-(\underline{A}) < \infty$ $(i = 1, 2)$. Also $(\psi_1 + \psi_2)^- \leq \psi_1^- + \psi_2^-$ (from the minimizing property of the Jordan form). Hence

$$(\psi_1 + \psi_2)^-(\underline{A}) < \infty \qquad (3.3.4)\ \cancel{(4)}$$

Also

$$(\psi_1 + \psi_2)^+(\underline{A}) + \psi_1^-(\underline{A}) + \psi_2^-(\underline{A}) = (\psi_1 + \psi_2)^-(\underline{A}) + \psi_1^+(\underline{A}) + \psi_2^+(\underline{A}) \qquad (3.3.5)\ \cancel{(5)}$$

by the equivalence criterion (2.4) ~~(4) of the preceding section.~~ Since $\psi_1^-(\underline{A})$ and $\psi_2^-(\underline{A})$ are both finite, we may subtract them from both sides of (5) to obtain

$$(\psi_1 + \psi_2)^+(\underline{A}) = (\psi_1 + \psi_2)^-(\underline{A}) + [\psi_1^+(\underline{A}) - \psi_1^-(\underline{A}) + \psi_2^+(\underline{A}) - \psi_2^-(\underline{A})]$$

$$\geq (\psi_1 + \psi_2)^-(\underline{A}), \qquad (3.3.6) ~~(6)~~$$

since (3) applies to ψ_1 and ψ_2. Thus (4) and (6) show that (3) is satisfied by $\psi_1 + \psi_2$. ~~This completes the proof.~~ □

We are now assured that the following definition is consistent.

__Definition:__ __Standard order__, $\succcurlyeq$, on the space of pseudomeasures, is the vector partial order whose positive cone is

$$\{\psi \mid \psi^+(\underline{A}) \geq \psi^-(\underline{A}) \text{ and } \psi^-(\underline{A}) < \infty\}. \qquad (3.3.7) ~~(7)~~$$

We ~~shall~~ use the notations $\succcurlyeq$ and $\geq$ to distinguish standard from narrow order, respectively; $\succ$ and $>$ are the corresponding strict inequality signs; the indifference sign $\sim$ will refer to indifference under __standard__ order only. (It is not needed for narrow order, since indifference coincides with equality there).

The positive cone (7) consists precisely of those pseudomeasures which are signed measures μ satisfying $\mu(\underline{A}) \geq 0$. It follows that a pseudomeasure is comparable to zero under standard order iff it is a signed measure.

~~Let us~~ We first verify the claim made above, that if μ and ν are __finite__ signed measures, the relation between them under standard order is the same as that given by the comparison of $\mu(\underline{A})$ and $\nu(\underline{A})$. Here μ and ν are identified, as usual, with the __pseudomeasures__ (μ^+, μ^-), (ν^+, ν^-), respectively. (In particular, the ordering of finite definite integrals is the same as the standard ordering of the corresponding indefinite integrals).

Theorem: Let μ and ν be finite signed measures on space $(\underline{A},\Sigma)$. Then $(\mu^+,\mu^-) \succcurlyeq (\nu^+,\nu^-)$ iff $\mu(\underline{A}) \geq \nu(\underline{A})$.

Proof: Since $\succcurlyeq$ is a vector partial order, $(\mu^+,\mu^-) \succcurlyeq (\nu^+,\nu^-)$ iff

$$(\mu^+ + \nu^-, \nu^+ + \mu^-) = (\mu^+,\mu^-) - (\nu^+,\nu^-) \succcurlyeq 0. \qquad (3.3.8)\ \cancel{(8)}$$

Let (λ^+,λ^-) be the Jordan form of $(\mu^+ + \nu^-, \nu^+ + \mu^-)$.

By the equivalence criterion,

$$\lambda^+ + \nu^+ + \mu^- = \lambda^- + \mu^+ + \nu^-. \qquad (3.3.9)\ (9)$$

Since all measures in (9) are finite, we obtain

$$\lambda^+(\underline{A}) - \lambda^-(\underline{A}) = [\mu^+(\underline{A}) - \mu^-(\underline{A})] - [\nu^+(\underline{A}) - \nu^-(\underline{A})] = \mu(\underline{A}) - \nu(\underline{A}). \qquad (3.3.10)\ (10)$$

Since (8) holds iff $\lambda^+(\underline{A}) - \lambda^-(\underline{A}) \geq 0$, ~~so~~ (10) completes the proof.

§10-12

Strict inequality and indifference take a simple form for standard order. The following results are immediate from the definitions.

Theorem: Let ψ be a pseudomeasure on $(\underline{A},\Sigma)$.

(i) $\psi \succ 0$ iff $\psi^+(\underline{A}) > \psi^-(\underline{A})$.

(ii) $\psi \sim 0$ iff $\psi^+(\underline{A}) = \psi^-(\underline{A})$ and these are finite.

(iii) ψ, 0 are not comparable iff $\psi^+(\underline{A}) = \psi^-(\underline{A}) = \infty$.

As with any vector partial order, we then have, for pseudomeasures ψ_1 and ψ_2, $\psi_1 > \psi_2$ iff $(\psi_1 - \psi_2) > 0$; $\psi_1 \sim \psi_2$ iff $(\psi_1 - \psi_2) \sim 0$; ψ_1, ψ_2 not comparable iff $\psi_1 - \psi_2$, 0 not comparable.

Theorem: Standard order extends narrow order.

Proof: Let $\psi_1 > \psi_2$ (narrow order); then $(\psi_1 - \psi_2) > 0$; ~~that is,~~ i.e., $(\psi_1 - \psi_2)^-(\underline{A}) = 0$, while $(\psi_1 - \psi_2)^+(\underline{A}) > 0$. Hence $(\psi_1 - \psi_2) > 0$, so that $\psi_1 > \psi_2$.

The corresponding result for indifference is trivial, since $\geq$ is antisymmetric. □

If Σ is not trivial, then this extension is proper; ~~that is,~~ i.e., there are ψ_1, ψ_2 which that are non-comparable under narrow order but comparable under standard order. An example is given in the proof that narrow order is incomplete. The same example shows that standard order is not anti-symmetric, except in the trivial case $\Sigma = \{\emptyset, \underline{A}\}$.

Theorem: Standard order is incomplete, except when Σ is a finite ~~sigma~~ σ-field.

Proof: Let Σ be finite. Then it is generated by a finite partition $\{A_1, \ldots, A_n\}$. If μ is a ~~sigma~~ σ-finite measure, then $\mu(A_{\underline{i}})$ is finite for all $\underline{i} = 1\ldots\underline{n}$, so μ is in fact finite. Hence $\psi^+(\underline{A})$, $\psi^-(\underline{A})$ are both finite, for all pseudomeasures ψ, and all pairs are comparable: $\geq$ is complete.

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Conversely, let Σ be infinite. Let $\mathcal{F} \subseteq \Sigma$ be infinite and countable, and let $\mathcal{G}$ consist of all $\mathcal{F}$-sets, ~~together~~ along with their complements. For each $a \in A$, let $E(a)$ be the intersection of all $\mathcal{G}$-sets to which a belongs. It may be shown that these sets $E(a)$ form an infinite measurable partition. (Details are omitted).

There is thus an infinite sequence $(E_1, E_2, \ldots)$ of non-empty, measurable, mutually disjoint sets. Choose a point $a_n \in E_n$ for all $n = 1, 2, \ldots$, and define the measures μ, ν by:

$\mu(E)$ = number of points $a_n \in E$ for which n is odd, and

$\nu(E)$ = number of points $a_n \in E$ for which n is even,

for all measurable E. Thus $\mu(E_{2m-1}) = \nu(E_{2m}) = 1$, and $\mu(E_{2m}) = \nu(E_{2m-1}) = 0$, all $m = 1, 2, 3, \ldots$. also μ and ν are ~~sigma~~ σ-finite, infinite, and mutually singular. Hence the pseudo-measure (μ, ν) is not comparable to 0 under standard order: $\geq$ is incomplete. □

The case where Σ is finite is not very interesting from our present point of view, ~~For~~ because in this case all pseudomeasures are finite signed measures, and ~~, in fact,~~ Ψ is isomorphic to ordinary n-space for some $n = 0, 1, \ldots$, as we have noted above. Thus we may say: whenever pseudomeasures are interesting, standard order is incomplete.

(B)

Applications

We ~~shall~~ consider here only the more important category of applications, i.e. the use of pseudomeasures to represent preferences. Let $\underline{P}$ be a set of conceivable alternative options in a certain situation, and let $\succcurlyeq$ be a preference ordering on $\underline{P}$. ($\succcurlyeq$ is a partial order, i.e. ~~that is~~, a transitive, reflexive relation). We represent this ordering by a pseudomeasure-valued utility function $\underline{p} \to \psi(\underline{p})$, mapping $\underline{P}$ into Ψ, the set of pseudomeasures over some space $(\underline{A},\Sigma)$. (Set $\underline{A}$ might be completely unconnected with $\underline{P}$, but usually there is some connection that ~~which~~ makes the representation "natural".) That is, for any two options $\underline{p}_1$, $\underline{p}_2 \in \underline{P}$, we have

$$\underline{p}_1 \succcurlyeq \underline{p}_2 \quad \text{iff} \quad \psi(\underline{p}_1) \succcurlyeq \psi(\underline{p}_2) \qquad \text{(3.3.11)} \; \cancel{(11)}$$

Here "$\succcurlyeq$" on the left is the preference ordering, while "$\succcurlyeq$" on the right is standard ordering on the space Ψ.[10]

The "size" comparison between two given pseudomeasures is usually quite simple to make. For the great bulk of applications in this book (and this will probably be true in further applications as well), $\psi(p)$ takes the form of an integral, in which either the integrand or the measure does not depend on $\underline{p}$. Thus, in the fixed-integrand case, we need only worry about whether a statement of the following form is true:

$$\int \underline{f}\, d\mu_1 \succcurlyeq \int \underline{f}\, d\mu_2 \qquad \text{(3.3.12)} \; \cancel{(12)}$$

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$\succcurlyeq$

(All indefinite integrals are over space A).

In the fixed-measure case, the relevant statements are all of the form

$$\int f_1 \, d\mu \geq \int f_2 \, d\mu. \qquad \text{(3.3.13)} \quad \cancel{(13)}$$

In turn, both ~~of~~ these statements are logically equivalent to a statement of the form

$$\int f \, d\mu \succcurlyeq 0. \qquad \text{(3.3.14)} \quad \cancel{(14)}$$

To go from (13) to (14), let $f = f_1 - f_2$. To go from (12) to (14), let $\mu = \mu_1 - \mu_2$. (If μ_1, μ_2 are both infinite measures, interpret $\mu_1 - \mu_2$ as the pseudomeasure (μ_1, μ_2).) If μ in (14) should turn out to be a signed measure, ~~then~~ the following simple but important result shows that the "size" comparison of pseudomeasures reduces to the evaluation of an ordinary definite integral.

Theorem: (standard integral theorem) Let μ be a ~~sigma~~ σ-finite signed measure on space (A, Σ), and let $f: A \to$ reals be measurable. Then (14) — in the sense of standard order — is true iff the definite integral

$$\int_A f \, d\mu \qquad \text{(3.3.15)} \quad \cancel{(15)}$$

is well-defined and non-negative.

Proof: The Jordan form of $\psi = \int f \, d\mu$ is

$$(\psi^+, \psi^-) = \left(\int f^+ \, d\mu^+ + \int f^- \, d\mu^-, \int f^- \, d\mu^+ + \int f^+ \, d\mu^- \right).$$

Relation (14) is true iff $\psi^+(A) \geq \psi^-(A) < \infty$, i.e., iff the double inequality

$$\int_A f^+ \, d\mu^+ + \int_A f^- \, d\mu^- \geq \int_A f^- \, d\mu^+ + \int_A f^+ \, d\mu^- < \infty$$

holds. But this is precisely the condition for (15) to be well-defined and non-negative. $\square$

Consider the procedure of Ramsey in his now-famous article.[11] Here each conceivable policy *p* is a time-path of consumption and labor over the entire positive half-line. The total utility resulting from policy *p* has the form

$$\int_0^\infty g(p,t)\,dt, \tag{3.3.16}$$

where $g(p,t)$ is the momentary utility from consumption net of the disutility from labor at instant *t* under policy *p*. (We need not be concerned with the exact form of this function). The trouble is that (16) will diverge for good policies. Accordingly, Ramsey uses, not (16), but

$$\int_0^\infty [b - g(p,t)]\,dt \tag{3.3.17}$$

as the objective function. Here b is "bliss", the highest attainable momentary net utility level; (17) is the shortfall from constant bliss, and is to be minimized. The integral is finite except for very poor policies, which may be ignored. Preference is now represented by an ordinary real-valued utility function, viz. ~~namely~~, the negative of (17).

We now treat the same problem by pseudomeasures. Let (A,Σ) be the positive half-line with Borel field. The utility assigned to policy p is the pseudomeasure

$$\psi(p) = \int g(p,t)\,dt.$$

("dt" of course refers to Lebesgue measure on the positive half-line). For two policies, p_1 and p_2, we then have $p_1 \succeq p_2$ iff

$$\int g(p_1,t)\,dt \succeq \int g(p_2,t)\,dt$$

(standard order). By the standard integral theorem, this is true iff the definite integral relation

$$\int_0^\infty [g(p_1,t) - g(p_2,t)]\,dt \geq 0 \qquad (3.3.18)\ ~~(18)~~$$

holds, the integral being well-defined. It is easy to see that (18) determines exactly the same ordering among policies as does (17), with the minor exception that (17) does not discriminate adequately among alternative very poor policies for which it equals $+\infty$.

Thus the Ramsey approach is in this case essentially the same as the pseudomeasure approach. What is gained by using the latter? First, ~~one~~ we avoid~~s~~ the slightly *ad hoc* "bliss" procedure, which may not be available for other problems. But, more important, ~~one has~~ we have a unified procedure, which works for multidimensional and abstract spaces, which works when the measure rather than the integrand varies (as in Bernoullian utility discussed below), etc.

~~Let us~~ We also sharply distinguish the Ramsey approach from the "overtaking" approach ~~which~~ that grew out of it. "Overtaking" depends essentially on the order or metric properties of the real line; the Ramsey approach does not. It turns out that, just as the Ramsey approach is a special case of standard ordering of pseudomeasures, the "overtaking" approach is a special case of extended ordering. This is ~~all~~ discussed below.

The following simple problem offers further insights into the use of standard order. Let (A, Σ, μ) be a probability measure space; ~~– that is~~ i.e., μ is a measure with $\mu(A) = 1$. Let $f: A \to$ reals be measurable, and let the definite integral

$$\int_A f \, d\mu$$

exist and be finite, with value c.[12] Consider the problem of minimizing

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$$v(x) = \int_A [f(a) - x]^2 \, \mu(da) \qquad \text{(3.3.19)} \; \cancel{(19)}$$

over real numbers x. It is well-known that the unique minimizer for this problem is $x = c$, provided the integral (19) is finite for all x. If (19) is infinite for some x, ~~then~~ it is infinite for all x, so that every real number is a minimizer. We now show that the use of pseudomeasures allows this proviso to be dropped.

Thus we ~~let us~~ now rewrite $v(x)$ as $\psi(x)$, and interpret (19) as the indefinite integral. There are two preliminary minor points to ~~take~~ note ~~of~~. First, we are minimizing, so "smaller" is "better"; but under standard ordering "larger" is "better" in an obvious sense. This difficulty may be remedied in either of two equivalent ways: ~~(i)~~ insert a minus sign ~~"−"~~ in front of (19) to convert it to a maximum problem, or ~~(ii)~~ use reverse standard ordering rather than (direct) standard ordering, defined by: $\psi_1 \geq \psi_2$ in the reverse sense iff $\psi_1 \leq \psi_2$ in the direct sense. We shall use the latter approach. Second, ~~one should~~ remember that standard order is, in general, not complete, so that there are two possible senses in which a solution may be optimal: it may be best, or it may be merely unsurpassed. In the following theorem the stronger of these two senses may be asserted.

g11-3

Theorem: Let μ, f, and c be as above. The problem of minimizing $\psi(x)$ over real numbers x has a unique best solution, viz. ~~namely,~~ $x = c$. ("Minimization" is understood in the sense of reverse standard ordering).

D

Proof: Let real $x \neq c$, and consider the definite integral

$$\int_A [(f(a) - x)^2 - (f(a) - c)^2]\mu(da)$$

$$= \int_A [2(c - x)f(a) + x^2 - c^2]\mu(da) = (c - x)^2 > 0.$$

Since this is well-defined and positive, it follows that the indefinite integral satisfies

$$\int [(f(a) - x)^2 - (f(a) - c)^2]\mu(da) \succ 0$$

(standard order), by the standard integral theorem. Thus

$$\psi(x) \succ \psi(c) \qquad \text{(3.3.20)}$$

(standard order), for all $x \neq c$. Thus c is best under reverse standard ordering. Because indifference is precluded in (20), c is the unique number with this property. ||| □

In probability terms, then, we may say: the second moment of f about x is uniquely minimized when x = expectation f (even if the second moment is infinite under conventional calculation).

Bernoullian Utility under Standard Order

Let (A,Σ) be, as usual, a measurable space fixed throughout the discussion, and let P be the set of all probability measures on this space. We are concerned with preference orderings over P. The modern discussion of this subject arises from the observation[13] that, under certain quite plausible assumptions concerning the preference ordering, $\succ$, a rather strong conclusion could be drawn (for Σ finite): ~~There exists~~ a measurable function $u: A \to$ reals such that the mapping $p \to \int_A u\,dp$, which assigns to each $p \in \mathcal{P}$ the expectation of u with respect to p, is a utility function ~~which~~ that represents $\succ$. This is the expected utility theorem.

This result has been generalized in various ways, but these generalizations always run up against the obstacle that the integral $\int_A u\,dp$ must be well-defined and finite. In practice this means either that u is bounded, or the p's must be restricted to a small subset of $\mathcal{P}$ (once ~~one~~ we goes to an infinite ~~sigma~~ σ-field; if Σ is finite, u is automatically bounded).[14]

Both ~~of~~ these restrictions are objectionable. The restriction of $\mathcal{P}$ to finitely concentrated probabilities (see below) simply does not allow enough scope. The objections to bounded u require more discussion:

Consider the following "Archimedean" postulate. Let a_1, a_2, $a_3 \in A$, with $a_1 \succ a_2 \succ a_3$; then ~~there exists~~ a number x, $0 < x < 1$, such that

$$[(1 - x)a_1 + x\,a_3] \succ a_2. \qquad \text{(3.3.21)} \ \cancel{(21)}$$

(Here "a_i" refers to the probability measure with all mass simply concentrated on point a_i, $i = 1, 2, 3$, and (21) states that some probability mixture of a_1 and a_3 is preferred to a_2.)

Suppose a_1 and a_2 are situations ~~which~~ differ ing only in some trivial respect ~~— say~~ (e.g., having this morning's egg boiled for 3 minutes vs. 3 minutes and 1 second, while a_3 is a horrendous situation such as a world pandemic or thermonuclear war. ~~One~~ We may argue that (21) is still satisfied by some x ~~which is~~ very very close to 0 in value. The point is controversial.

11-4

Now suppose the expected utility condition holds, with a function u that is bounded below. We claim ~~that~~ this has a consequence ~~which~~ that is less plausible than (21) by an order of magnitude. Without loss of generality let $u(a_1) = 1$, $u(a_2) = 0$, and let $-M$ be a lower bound for u, where M is a large positive real number. Then, for any choice of a_3, (21) is satisfied by $x = 1/(M + 2)$, since for the left side of (21) then has utility at least equal to $1/(M + 2) > 0$. That is, the mixture proportions, x and $1 - x$, may be chosen in advance of knowing a_3, no matter how horrible. The Archimedean postulate allows x to depend on a_3, so that progressively more horrible situations may be counterbalanced by being given progressively less weight.

There is a similar implausibility argument for u bounded above. Start with the "dual" Archimedean postulate ~~which~~ that replaces all ">" signs by "<", let a_1, a_2 differ trivially as above, and let a_3 be some highly desirable situation such as universal salvation or utopia.

In general, bounded utility appears to characterize orderings with a certain pettifogging quality, in which there is no Pascalian wager, no Faustian aspiration, no Promethean ambition. To exclude these preferences would be to exclude the values of many makers and shapers of history, not all of whom are irrational.

What these thoughts amount to is this: Any axiom system ~~which~~ impl~~ies~~ying that "rational" preference orderings satisfy the expected utility condition with bounded utility function is simply too restrictive.

But if the boundedness restriction on u is removed, what sense is one to make of the integral $\int_A u\,dp$? Our recommendation should not be too surprising: Reinterpret this an an *indefinite* integral, and let size ordering among these entities be given by *standard ordering* of pseudomeasures.

But why should one do this? Just as the ordinary expected utility condition needs justification, so too does this standard ordering condition. We now give a set of axioms ~~which~~ that implies it. These axioms have about the same general level of plausibility as those in customary use — even a bit more, ~~plausibility in fact,~~ since they are weakened to the point where they do not imply that *u* is bounded.

The main limitation imposed is that Σ be generated by a *countable partition* of *A*. This limitation is regrettable, but still allows the main point to come through: ~~There~~ exists an axiomatic basis for the use of unbounded utility functions and standard ordering in the treatment of uncertainty.[15]

Without real loss of generality, we may assume that each element in the partition generating Σ is singleton. Thus we get: *A* countable, and Σ = all subsets of *A*. This is the space on which we work. Each probability measure *p* on (*A*,Σ) is completely determined by its values on the singleton sets {a}; in fact, it will sometimes be convenient in what follows to think of probabilities as point functions (with domain A), and we write p(a) instead of the technically correct p({a}).

Probability measure p is said to be finitely concentrated iff there is a finite set $E \subseteq A$ such that $p(A \setminus E) = 0$; ~~in other words,~~ i.e., $p(a) = 0$ for all but a finite number of points $a \in A$. The set of all probabilities will be denoted by $\mathcal{P}$, as above, while the subset of finitely concentrated probabilities will be denoted by $\mathcal{F}$.

In axiom 4 the following concept is used: Real sequence $x_1, x_2, \ldots$ is monotone iff it is either non-increasing or non-decreasing: either $x_1 \geq x_2 \geq \ldots$, or $x_1 \leq x_2 \leq \ldots$.

Axioms concerning partial order $\succ$ on $\mathcal{P}$ follow:

Axiom 1: Any two finitely concentrated probabilities are comparable.

Axiom 2: Let $p_1, q_1 \in \mathcal{P}$, let $p_2, q_2 \in \mathcal{F}$, with $p_1 \sim q_1$ and $p_2 \succ q_2$; let x be a number, $0 < x < 1$; then

$$[(1 - x)p_1 + x\, p_2] \succ [(1 - x)q_1 + x\, q_2]. \quad (22)$$

(3.3.22)

Axiom 3: Let $p_1, q_1 \in \mathcal{P}$ with $p_1 \succ q_1$; then ~~there~~ exist $p_2, q_2 \in \mathcal{F}$, and a number x, $0 < x < 1$, such that $p_2 \prec q_2$ and (22) is true.

Axiom 4: Let $p, q \in \mathcal{P}$; let p_k, q_k, $k = 1, 2, \ldots$, be two sequences of finitely concentrated probabilities such that, for all $a \in A$, the three sequences $(p_k(a))$, $(q_k(a))$, and $(p_k(a) - q_k(a))$,

$\underline{k} = 1, 2, \ldots$, are monotone, and such that, for all $a \in A$,

$$\lim_{k \to \infty} p_k(a) = p(a), \quad \cancel{\text{and}} \quad \lim_{k \to \infty} q_k(a) = q(a),$$

and such that, for all $k = 1, 2, \ldots$, $p_k \succ q_k$; then it is false that $q \succ p$.

Axiom 5: Let $\succ'$ be any partial order on P ~~which~~ that satisfies axioms 1 ~~through~~ 4 and ~~which~~ that extends $\succ$:

$p \succ q$ implies $p \succ' q$, and

$p \sim q$ implies $p \sim' q$;

then $\succ$ and $\succ'$ are identical.

Axiom 1 is a weak completeness axiom. Axiom 2 is a form of the strong independence axiom, and asserts, roughly, that mixing in a pair of indifferent probabilities does not disturb order of preference. Axiom 3 is a weak axiom ~~which~~ and asserts, roughly, that for any $p_1 \succ q_1$, ~~one~~ we can find finitely concentrated $p_2 \prec q_2$ for which the preference intensity is not infinitely stronger than the original. Axiom 4 is an Archimedean axiom of sorts, and asserts that, under certain conditions, if sequences p_k, q_k converge to p, q respectively, it cannot happen that preferences between p_k and q_k all run in one direction and preference between p, q runs in the opposite direction. (The monotonicity clause in axiom 4 has no intuitive appeal in itself, but note that its insertion weakens the axiom, and thereby makes axiom 4 more plausible in the logical sense). Finally, axiom 5, like axiom 1, is a weak completeness axiom, and asserts that $\succ$ has maximal comparability in the class of partial orders satisfying axioms 1 ~~through~~ 4. (The assumption that any two probabilities are comparable — which we do not make — would imply both axioms 1 and 5).

The following lemma asserts that axioms 1 ~~through~~ 4 alone guarantee the existence of a function u ~~which~~ that provides a "nonfaithful" representation of $\succ$ in the sense of Aumann.[16]

Lemma: Let $\mathbb{P}$ be the set of all probability measures on (A,Σ), where A is countable and $\Sigma =$ all subsets of A; let $\succ$ be a partial ordering on $\mathbb{P}$ satisfying axioms 1 ~~through~~ 4.

Then ~~there~~ exists a function $u: A \to$ reals such that, for all $p, q \in \mathbb{P}$,

$$p \succ q \text{ implies } \int u\, dp \succ \int u\, dq, \qquad (3.3.23)\ ~~(23)~~$$

~~and~~

$$p \sim q \text{ implies } \int u\, dp \sim \int u\, dq. \qquad (3.3.24)\ ~~(24)~~$$

(Here "$\succ$" and "$\sim$" on the left refer to the partial order on $\mathbb{P}$, while "$\succ$" and "$\sim$" on the right refer to standard order on the space of pseudomeasures over (A,Σ).)

Proof: Let us first prove another "Archimedean" condition: If $q, p, p' \in \mathbb{F}$ (i.e. they are finitely concentrated), and $p' \succ q \succ p$, then ~~there~~ exists a number x, $0 < x < 1$, such that

$$(1 - x)p + xp' \prec q. \qquad (3.3.25)\ ~~(25)~~$$

To show this, define the two sequences p_k, q_k, $k = 1, 2, \ldots$, by:

$$p_k = \left[\frac{k}{k+1}\right]p + \left[\frac{1}{k+1}\right]p'$$

[k/(k+1)]p + [1/(k+1)]p'

and

$$q_k = q.$$

The p_k, q_k are all finitely concentrated; they converge pointwise to p, q, respectively, and the monotonicity clause of axiom 4 is satisfied. But the conclusion of axiom 4 is false, since $q > p$; hence the remaining premise of axiom 4 must be false, so that there exists a k_o for which $p_{k_o} > q_{k_o}$ is false. By axiom 1, it follows that $p_{k_o} \lesssim q_{k_o}$. There are two cases:

If $p_{k_o} < q_{k_o}$, then (25) is verified with $x = 1/(k_o + 1)$, and

If $p_{k_o} \sim q_{k_o}$, we apply axiom 2 to obtain

$$\left[\left(\frac{k_o + 1}{k_o + 2}\right)p_{k_o} + \left(\frac{1}{k_o+2}\right)p\right] < \left[\left(\frac{k_o + 1}{k_o + 2}\right)q_{k_o} + \left(\frac{1}{k_o+2}\right)q\right],$$

which verifies (25) with $x = 1/(k_o + 2)$.

By a similar argument (interchanging the rôles of p and p') we can show the existence of a number x such that (25') is true, where (25') is obtained from (25) by substituting ">" for "<".

Axioms 1 and 2, ~~together~~ along with conclusions (25) and (25') and Theorem 8.2 of Fishburn[17], imply the existence of a function u: $A \to$ reals, such that, for all *finitely concentrated* p, q, we have

$$p \succ q \text{ iff } \int_A u\,dp \geq \int_A u\,dq. \tag{3.3.26}$$

We will show that this u satisfies (23) and (24).

To prove (23), let $p, q \in \mathcal{P}$ with $p \succ q$, and define the function $f: A \to$ reals by

$$f(a) = u(a)[q(a) - p(a)]. \tag{3.3.27}$$

The hardest part of the proof will be to show that the sum of the **positive** terms of $f(a)$, summed over $a \in A$, is **finite**. Arguing by contradiction, suppose that the sum of $f^+(a)$ is $+\infty$. Then there exist a number $\delta > 0$ and an enumeration $a_0, a_1, \ldots$ of the points of A such that

$$f(a_0) + f(a_1) + \ldots + f(a_n) \geq \delta \tag{3.3.28}$$

for all $n = 0, 1, 2, \ldots$ (let a_0 be any point with $f(a_0) > 0$, and let $\delta = f(a_0)$; then enumerate the remaining positive and negative terms and choose enough positive terms to overbalance the first negative term, enough positive terms after that to overbalance the next negative term, etc.)

Next define a sequence (p_k), $k = 1, 2, \ldots$, of finitely concentrated probabilities as follows:

$$\begin{aligned} p_k(a_0) &= p(a_0) + [p(a_{k+1}) + p(a_{k+2}) + \ldots], \\ p_k(a_i) &= p(a_i) \quad \text{for } i = 1, \ldots, k, \\ p_k(a_i) &= 0 \quad \text{for } i > k. \end{aligned} \tag{3.3.29}$$

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A sequence (q_k), $k = 1, 2, \ldots$, is defined similarly, with q taking the rôle of p in ~~(31)~~ (29).

We claim that $p_k \succ q_k$ for an infinite number of k-values. For suppose this were false; then, by axiom 1, $q_k \succ p_k$ for all k past a certain value k_o. Furthermore, we have

$$\lim_{k\to\infty} p_k(a) = p(a), \quad \text{and} \quad \lim_{k\to\infty} q_k(a) = q(a)$$

for all $a \in A$. Also, each of the three sequences $(p_k(a))$, $(q_k(a))$, and $(p_k(a) - q_k(a))$, $k = 1, 2, \ldots$, is monotone for all $a \in A$ except possibly for $(p_k(a_\theta) - q_k(a_\theta))$. (Since, for $a = a_i$, $i \neq 0$, each of these sequences is 0 for $k < i$ and is constant for $k \geq i$). As for a_θ, we note that any real sequence has a monotone subsequence; hence there is a subsequence that satisfies the monotonicity clause for all $a \in A$, including a_θ. Axiom 4 now implies that $p \succ q$ is false: contradiction. Hence, indeed, $p_k \succ q_k$ infinitely often. Let $k_1 < k_2 < \ldots$ be the k-values for which this is true.

a_0 zero

(11-7)

Now apply (26) to each such k_n. Evaluating the integrals in (26) — which are just finite sums — and substituting from (29) and then (27), we obtain

summation

$$f(a_\theta) + f(a_1) + \ldots + f(a_{k_n}) + u(a_\theta)\left[\sum_{i=k_n+1}^{\infty} (q(a_i) - p(a_i))\right] \leq 0, \qquad \text{(3.3.30)} \; ~~(30)~~$$

for $n = 1, 2, \ldots$.

This, however, contradicts (28), for as $n \to \infty$ the sum of the f-terms remains $\geq \delta > 0$, while the bracketed expression in (30) converges to 0; hence the left side of (30) is positive for sufficiently large n.

We have now achieved our contradition, and conclude that, indeed, the sum of the positive terms of $f(a)$ must be finite.

Next, consider the definite integral

$$\int_A u \, d(q-p) = \left[\int_A u^+ \, d(q-p)^+ + \int_A u^- \, d(q-p)^-\right]$$

$$- \left[\int_A u^- \, d(q-p)^+ + \int_A u^+ \, d(q-p)^-\right].$$

The sum of the first two integrals on the right equals the sum of $f^+(a)$ over $a \in A$. Since this is finite, the integral is well-defined; furthermore, the sum of $f(a)$ over $a \in A$ converges to the same number (possibly $-\infty$) regardless of the order of summation, and this number is the value of the integral. We will now show that this value is non-positive.

Let $a_0, a_1, \ldots,$ be any enumeration of the points of A, and define p_k, q_k as in (29). The argument above shows that $p_k > q_k$ infinitely often; hence, via (26) again, we obtain (30). As $n \to \infty$, the left side of (30) converges to the sum of $f(a)$. Hence this is ≤ 0. We have proved that l.c. p

$$\int_A u\, d(q-\cancel{q}p) \leq 0 . \qquad \text{(3.3.31)} \ \cancel{(31)}$$

In turn, this implies

$$\int u\, dp \succsim \int u\, dq \qquad \text{(3.3.32)} \ \cancel{(32)}$$

(standard order), by the standard integral theorem. To establish (23), we must strengthen this to strict preference ~~inequality.~~ By axiom 3, there exist p', $q' \in F$ and a number x, $0 < x < 1$, with

$$p' < q' \qquad \text{(3.3.33)} \ \cancel{(33)}$$

and

$$(1-x)p + xp' > (1-x)q + xq' . \qquad \text{(3.3.34)} \ \cancel{(34)}$$

Relations (33) and (26) yield

$$\int_A u\, dp' < \int_A u\, dq' , \qquad \text{(3.3.35)} \ \cancel{(35)}$$

while the same argument that led from $p > q$ to (31) ~~(32)~~ now leads from (34) to

$$\int_A u\, d[(1-x)q + xq' - (1-x)p - xp'] \leq 0 . \qquad \text{(3.3.36)} \ (36)$$

Inequalities (36) and (35) in turn yield the respective inequalities:

$$\int_A u\, d(q-p) \leq \left[\frac{x}{1-x}\right] \int_A u\, d(p'-q') < 0 . \qquad \text{(3.3.37)} \ \cancel{(37)}$$

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Thus the integral in (31) is actually negative, which strengthens (32) to strict ~~inequality.~~ preference. This proves (23).

18 The proof of (24) now follows easily. Let $p \sim q$. If u is constant then (24) is trivial. If u is not constant, then ~~there~~ exist $p', q' \in \mathscr{F}$ with $p' > q'$. For any x, $0 < x < 1$, it then follows by axiom 2 that (34) is true. The argument above then yields the left inequality in (37). Since x can be arbitrarily close to 0, (31) must be true, which yields (32). Interchanging the rôles of p and q, the opposite inequality must also hold. Hence

(break only at ~)

$$\int u\,dp \sim \int u\,dq \quad \text{(standard order)}.$$

~~This completes the proof.~~

24 We now come to the main result.

Theorem: Let $\mathscr{P}$ be the set of all probability measures on (A,Σ), where A is countable and Σ = all subsets of A; let $\gtrsim$ be a partial ordering on $\mathscr{P}$. Then each of the following conditions implies the other:

18 (i) $\gtrsim$ satisfies axioms 1 ~~through~~ – 5;

(align (i) right)

(ii) ~~There~~ exists a function $u: A \to$ reals such that, for all $p, q \in \mathscr{P}$,

$$p \gtrsim q \text{ iff } \int u\,dp \gtrsim \int u\,dq. \qquad (3.3.38)$$

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(Here "$\succcurlyeq$" on the left refers to the partial order on $\mathcal{P}$; on the right it refers to standard order on the space of pseudomeasures over (A,Σ)).

Proof: Let function u satisfy (38); we must show that $\succcurlyeq$ on $\mathcal{P}$ satisfies axioms 1 ~~through~~ to 5.

If p, $q \in \mathcal{F}$, ~~then~~ the definite integral $\int_A u\, d(p-q)$ is well-defined; hence $\int u\, dp$, $\int u\, dq$ are comparable under standard order, by the standard integral theorem; hence p, q are comparable by (38). This proves axiom 1.

Let p_1, q_1, p_2, q_2 satisfy the premises of axiom 2, so that, by (38),

$$\int u\, dp_1 \sim \int u\, dq_1$$

and

$$\int u\, dp_2 \succ \int u\, dq_2$$

(standard order).

By elementary pseudomeasure operations we find, for $0 < x < 1$, that

$$\int u\, d[(1-x)p_1 + xp_2] \succ \int u\, d[(1-x)q_1 + xq_2].$$

This, with (38), yields (22), and proves axiom 2.

Let $p_1 \succ q_1$; it follows from (38) that u cannot be constant, for this would imply universal indifference; hence ~~there~~ exist p_2, $q_2 \in \mathcal{F}$ with $p_2 \prec q_2$. Choose any positive x less than

$$\bar{x} = \int_A u\,d(p_1-q_1) \Big/ \left[\int_A u\,d(p_1-q_1) + \int_A u\,d(q_2-p_2)\right]. \qquad (3.3.39)\ \cancel{(39)}$$

(Both definite integrals in (39) are > 0, and $\int_A u\,d(q_2-p_2) < \infty$. If $\int_A u\,d(p_1-q_1) = \infty$, interpret $\bar{x}$ as 1; in any case $0 < \bar{x} \le 1$, so that x exists.) One easily verifies that

$$\int_A u\,d[(1-x)p_1 + xp_2 - (1-x)q_1 - xq_2] > 0$$

This, with (38) and the standard integral theorem, yields (22) and proves axiom 3.

Establishing axiom 4 is the difficult part of the proof. We ~~shall~~ argue by contradiction. Let p, q, and sequences p_k, q_k, $k = 1, 2, \ldots$, satisfy all the premises of axiom 4, but also let $q > p$. For each k define the function $f_k: A \to$ reals by $f_k(a) = u(a)[q_k(a) - p_k(a)]$, and similarly define f as in (27). Let μ be counting ~~enumeration~~ measure on (A,Σ), so that integration of f_k with respect to μ is the same as summation of $f_k(a)$ over $a \in A$, which in turn is the same as integration of u with respect to signed measure $q_k - p_k$. Thus

$$\int_A f_k\,d\mu = \int_A u\,d(q_k-p_k) \qquad (3.3.40)\ \cancel{(40)}$$

$k = 1, 2, \ldots$. A similar relation holds for f, p, and q (just drop the subscript "k" in (40)).

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We shall deduce the contradiction

$$0 < \int_A f\, d\mu = \int_A (\liminf_{k\to\infty} f_k)\, d\mu \leq \liminf_{k\to\infty} \int_A f_k\, d\mu \leq 0. \qquad (3.3.41)\ \text{~~(41)~~}$$

The equality in (41) arises from the fact that

$$\lim_{k\to\infty} f_k(a) = f(a)$$

for all $a \in A$, so that the integrands are equal. The left inequality in (41) arises from the fact that $q > p$, ~~together~~ along with (38), the standard integral theorem, and (40) without subscript "k". The right inequality in (41) arises from the fact that, for each k, $p_k > q_k$, so that, by the same argument,

do not break

$$\int_A f_k\, d\mu < 0$$

for all $k = 1, 2, \ldots$.

This leaves the middle inequality of (41) to be verified. This is the conclusion of Fatou's lemma, and may be asserted **if** we can show the existence of a function g: $A \to$ reals such that $f_k \geq g$ for all k, and such that

$$\int_A g\, d\mu > -\infty. \qquad (3.3.42)\ \text{~~(42)~~}$$

We construct g as follows. Define the two subsets of A:

$$E = \{a \mid p_1(a) \neq q_1(a)\}, \quad N = \{a \mid f(a) < 0\},$$

and let

$$g(a) = -|u(a)| \qquad \text{if } a \in E,$$

$$g(a) = f(a) \qquad \text{if } a \in N\backslash E, \qquad (3.3.43)\ \ ~~(43)~~$$

$$g(a) = 0 \qquad \text{if } a \in A\backslash(N \cup E).$$

We ~~Let us~~ first verify that, for each $a \in A$, we have

$$f_k(a) \geq g(a), \qquad (3.3.44)\ \ ~~(44)~~$$

$k = 1, 2, \ldots$. This is obvious for $a \in E$, since p_k, q_k are probabilities. For $a \notin E$, we utilize the fact that $(p_k(a) - q_k(a))$, $k = 1, 2, \ldots$, is a monotone sequence. This implies that $f_k(a)$, $k = 1, 2, \ldots$, is a monotone sequence for each point a. Since $f_1(a) = 0$ for $a \notin E$, it follows that $f_k(a)$ lies between 0 and the limit $f(a)$ for each k. For $a \in N\backslash E$, we have $f(a) < 0$, so that (44) follows from (43). For $a \in A\backslash(N \cup E)$ we have $f(a) \geq 0$, so that $f_k(a) \geq 0$, again verifying (44).

Finally, we ~~let us~~ verify (42), which is true iff the sum of the negative terms of $g(a)$ over $a \in A$ is finite. The set E is finite, since p_1, q_1 are finitely concentrated. The set $A\backslash(N \cup E)$ makes no contribution to the sum. On the set $N\backslash E$ we have $g = f$, but the sum of the negative terms of $f(a)$ is finite by the left inequality in (41). Thus (42) is verified. We may now assert Fatou's lemma, yielding the contradiction (41). This proves axiom 4.

Let $\succsim'$ be another partial order on $\mathcal{P}$ satisfying the premises of axiom 5. Then $\succsim'$ satisfies axioms 1 ~~through~~ to 4; hence, by the preceding lemma, ~~there~~ exists a function u': $A \to$ reals such that, for all p, $q \in \mathcal{P}$,

$$\text{if } p \succsim' q \text{ then } \int u'\,dp \geq \int u'\,dq \qquad \text{(3.3.45)} \quad \cancel{(45)}$$

(standard order). Furthermore, if p, $q \in \mathcal{F}$, then the "if... then" of (45) may be strengthened to "if and only if", by axiom 1. ~~(26)~~ ~~(40)~~.

Relation $\succsim'$ extends $\succsim$, and $\succsim$ is complete on $\mathcal{F}$ (proof of axiom 1 above). It follows that $\succsim'$ and $\succsim$ coincide when restricted to $\mathcal{F}$. u and u' are then two Bernoullian utilities representing the same ordering on $\mathcal{F}$. It follows that u' is a positive affine transformation of u: ~~There~~ exist real numbers x, y, with $x > 0$, such that

$$u'(a) = xu(a) + y$$

for all $a \in A$.[18] But then, for *any* p, $q \in \mathcal{P}$ we have

$$\int_A u'\,d(p-q) = x\int_A u\,d(p-q) + \int_A y\,d(p-q)$$

$$= x\int_A u\,d(p-q)$$

(provided the left-~~hand~~ integral is well-defined). It follows by the standard integral theorem that the conclusion in (45) implies gbl 10

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$$\int u\, dp > \int u\, dq \qquad (3.3.46)\ \cancel{(46)}$$

(standard order). In turn, (46) implies $p \succ q$, by (38), which in turn implies $p \succ' q$, since $\succ'$ extends $\succ$. This, with (45), closes a circle of implications, and shows that, for all $p, q \in \mathcal{P}$,

$$p \succ' q \text{ iff } p \succ q.$$

This proves axiom 5.

18 Half ~~of~~ the proof is now complete; the other half now follows rapidly. Let $\succ$ satisfy axioms 1 ~~through~~ 5. By the preceding lemma, there exists a function $u: A \rightarrow$ reals "non-faithfully" representing $\succ$ by (23) and (24). Let $\succ'$ be the partial order on $\mathcal{P}$ determined by u according to (38). We show that $\succ'$ satisfies the premises of axiom 5. By the first half of this proof, $\succ'$ satisfies axioms 1 ~~through~~ 4. Furthermore, if $p \succ q$, then

$$\int u\, dp > \int u\, dq$$

by (23), which in turn implies $p \succ' q$, by (38); and if $p \sim q$, then

$$\int u\, dp \sim \int u\, dq,$$

by (24), which in turn implies $p \sim' q$, by (38). Thus $\succ'$ extends $\succ$. The premises of axiom 5 being satisfied, it follows that $\succ$ and $\succ'$ are identical. This completes the proof!

18 A few concluding comments follow. If universe set A is not merely countable but finite, ~~then~~ any utility function u must be bounded, and standard order reduces to the ordinary size comparison of definite integrals: We are back to the conventional expected utility condition. There is a corresponding simplification on the axiomatic front in this case; Every probability measure is now finitely concentrated ($\mathcal{F} = \mathcal{P}$), so that axiom 1 now asserts the completeness of $\succeq$. As mentioned above, this implies axiom 5 (Proof: if $\succeq$ is complete and $\succeq'$ extends $\succeq$, then $\succeq' = \succeq$!); hence axiom 5 may be discarded. Also axiom 2 now implies axiom 3 (Hint: Let $p_2 = q_1$, $q_2 = p_1$, $x = 1/3$; mix $\frac{1}{2}(p_1 + q_1)$ into both sides of $p_1 \succ q_1$); hence axiom 3 may also be discarded. We are left with the conventional three axioms of completeness (axiom 1), strong independence (axiom 2), and Archimedes (axiom 4). (Exercise: Show that the monotonicity clause in axiom 4 may now be deleted to yield an axiom logically equivalent to the original).

use small fraction

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Going back to the general case, let $\succeq$ have a representation (38) with u unbounded. What conventional axioms will $\succeq$ not satisfy? Strong independence still stands, but $\succeq$ is definitely not complete. For there will be a sequence of points $a_1, a_2, \ldots$ in A (not necessarily exhaustive) such that $u(a_n) > 2^n$, all n (if u is unbounded above), or such that $u(a_n) < -2^n$, all n (if u is unbounded below); let $p(a_{2n}) = q(a_{2n-1}) = 2^{-n}$, all $n = 1, 2, \ldots$, with zero values elsewhere; then break only at comma

$$\int u\, dp, \quad \int u\, dq$$

are not comparable under standard order, so p, q are not comparable under $\succ$.

Furthermore, the Archimedean axioms cannot hold in full generality. Specifically, if u is unbounded above, we ~~let us~~ show that (25) cannot hold for every probability triple $p' \succ q \succ p$ (It does hold for p', p, $q \in \mathcal{F}$ as proved above). Take a sequence $a_1, a_2, \ldots$ with $u(a_n) > 2^n$, all n, and, say, $u(a_2) > u(a_1)$. Let $p'(a_n) = 2^{-n}$, all n (the "Petersburg" distribution), and let $q = a_2$, $p = a_1$; then $p' \succ q \succ p$, but we can ~~it is easy to~~ see that (25) is false for every x, $0 < x < 1$. Similarly, (25') is false for u unbounded below.

These considerations give a clue as to how one might set about modifying existing models that ~~which~~ are too strong in that they imply bounded utility. For example, in Arrow's model it would be interesting to see the effect of relaxing his "monotone continuity" assumption, which has an "Archimedean" flavor.[19]

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3.4. Extended Ordering of Pseudomeasures

The virtues of standard order may be summarized again as follows: (i) It resolves the blurring of preferability relations which arises when objective-function integrals are infinite; (ii) it extends the scope of comparability by admitting policies whose integrals are not well-defined; and,

~~finally,~~ (iii) when alternative policies have integrals, ~~which are~~ well-defined and finite, the standard ordering criterion reduces to the ordinary size comparison of definite integrals.

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The one disquieting property ~~it~~ standard order has is that, generally, standard ordering is incomplete, and this raises the question: Is it worthwhile to extend standard order~~, so as~~ to make more pairs of pseudomeasures comparable? The affirmative is based on the feeling that one should be able to compare any two options; the negative is based on the feeling that any filling in of the "gaps" left by standard order involves arbitrary decisions ~~which~~ lacking the intuitive appeal of standard order.

Let us examine these issues. First ~~of all~~, the order to be concerned with is not standard order *per se*, but is that which it induces on the set of feasible alternatives. That is, although standard order is not complete, it is conceivable that, in any "nonartificial" problem, for any pair of feasible alternatives p_1 and p_2, the corresponding pseudomeasures ψ_1 and ψ_2 are comparable. ~~Actually,~~ This is probably the case for most ~~the great majority of~~ problems using pseudomeasure evaluations. However, there are "natural" problems — even in classical location theory — for which noncomparability arises. (The Löschian problem on the unbounded plane is an example.)

Secondly, completeness is not crucial. From the point of view of the theory of choice, the ideal situation obtains when, for any of the range of problems under consideration, there is a unique best choice. For this to occur, it is in general neither necessary nor sufficient that the order be complete.

Consider a pseudomeasure ψ ~~which~~ that is not comparable to 0 under standard order, so that $\psi^{+}(A) = \psi^{-}(A) = \infty$. Let (P, N) be a Hahn decomposition for ψ. ~~One~~ We can think of ψ intuitively as an infinite positive mass placed on P, coupled with an infinite negative mass placed on N. One possible way of achieving comparability with 0 would be to "cancel" patches of negative mass in N against patches of positive mass in P, and to come up with some kind of "net" mass, which may be positive, negative, or zero. The problem is to determine the method of "matching up" the patches to be canceled. This involves some more-or-less arbitrary rule; but, if the space A has some structure in addition to its ~~sigma~~ σ-field Σ (in particular, if it has a metric) there are some fairly "natural" ways ~~in which~~ this can be done.

Consider, ~~for example~~ (e.g.) a space A with measurable partition $\{A_1, A_2, \ldots\}$, and measures μ, ν with values on these sets as follows:

	A_1	A_2	A_3	$A_4 \ldots$
μ	2	0	2	0 ...
ν	0	1	0	1 ...

(3.4.1) ~~(1)~~

Since μ and ν are both infinite and mutually singular, ~~hence~~ they are not comparable under standard order. Yet μ *seems* to be bigger.

Actually, if the partition is arbitrary this feeling is an illusion. By clumping A_2, A_4, A_6 together, etc., ~~one~~ we can make ν seem bigger. But if the partition is somehow naturally ordered as it stands, ~~then one~~ we may try cancelling 1 against 2 with a net bigness rating to μ. An example might be where A is the nonnegative real numbers, and the A_n are a sequence of intervals in natural order.

We ~~shall~~ now consider some orderings ~~which are~~ based on the principle just outlined. Unlike narrow and standard order, which are uniquely determined by (A,Σ), there will in general be many of these "extended orderings," and it is a matter of *ad hoc* judgment as to which, if any, ~~of these~~ ~~is~~ are to be considered "correct." The extended order is determined by an extension class, which we now define.

Definition: Let (A,Σ) be a measurable space. A collection of measurable sets $\mathcal{F}$ is an extension class iff

(i) for all $F_1, F_2 \in \mathcal{F}$, ~~there is~~ a set $F_3 \in \mathcal{F}$ exists, such that $F_1 \cup F_2 \subseteq F_3$; ~~and~~

(ii) there is a countable subcollection $\mathcal{F}'$ ~~which~~ that covers A (~~that is~~ i.e., $A = \cup\mathcal{F}'$).

We give some examples.

1. ~~(i)~~ The class consisting of the universe set A alone is an extension class.

2. ~~(ii)~~ More generally, any collection of measurable sets including A is an extension class.

(~~iii~~ 3) More interesting examples arise when $\underline{A}$ is a metric space (and all open and closed discs are measurable). The collection of all closed discs $\{a \mid d(a,a^o) \leq r\}$ is an extension class, as is ~~likewise~~ the collection of all open discs.

(~~iv~~ 4) Let $\underline{A}$ be the non-negative real line. The class of all sets of the form $\{a \mid 0 \leq a \leq a^o\}$, for $a^o \in A$, is an extension class.

One or two more preliminary concepts are needed.

Definition: Let ψ be a pseudomeasure on (A,Σ). The restricted domain of ψ is the class of all measurable sets $\underline{E}$ such that $\psi^+(E)$ and $\psi^-(E)$ are not both infinite.

We denote this class by Σ_ψ. Clearly Σ_ψ coincides with Σ iff at least one of the two measures ψ^+, ψ^- is finite, i.e., ~~which is to say~~, iff ψ is a signed measure.

Definition: The value function of pseudomeasure ψ has domain Σ_ψ, and to all $E \in \Sigma_\psi$ assigns the value $\psi^+(E) - \psi^-(E)$.

Without risk of confusion, we ~~shall~~ denote the value of the value function of ψ at the set $E \in \Sigma_\psi$ by the symbol $\psi(E)$. The latter is therefore an extended real number for any such $\underline{E}$. ~~Note that~~ when pseudomeasure ψ is in fact a signed measure, its value function is precisely that signed measure in the ordinary sense of the term: a countably additive function on the ~~sigma~~ σ-field Σ. In all other cases, however, the domain Σ_ψ of the value function is no longer a ~~sigma~~ σ-field.

We now come to the vector partial order on Ψ determined by an extension class $\mathcal{F}$. As with standard order, we first prove a preliminary result to guarantee the consistency of the forthcoming definition.

script F

Lemma: Let $\mathcal{F}$ be an extension class on measurable space (A, Σ). The following set of pseudomeasures is a convex cone: ~~the set of all pseudomeasures ψ satisfying~~

18 (i) $\mathcal{F} \subseteq \Sigma_\psi$; (3.4.2) ~~(2)~~

~~and~~

(ii) for all $\varepsilon > 0$ there is an $\mathcal{F}$-set F_ε such that, for all $\mathcal{F}$-sets F containing F_ε,

$$\psi(F) > -\varepsilon .$$ (3.4.3) ~~(3)~~

~~is a convex cone.~~

Proof: The zero pseudomeasure satisfies (2) and (3). Let ψ satisfy them, and let b be a positive real number. The restricted domains of ψ and $b\psi$ are the same, and, for all $E \in \Sigma_\psi$, $(b\psi)(E) = b\psi(E)$. Given $\varepsilon > 0$, choose $F_{(\varepsilon/b)}$; then for $\mathcal{F}$-set F containing this set we have $\psi(F) > -\varepsilon/b$, so that $(b\psi)(F) > -\varepsilon$. Hence $b\psi$ satisfies (2) and (3).

It remains to show that, if ψ_1 and ψ_2 satisfy (2) and (3), so does $\psi_1 + \psi_2$. First ~~of all~~, ψ_1^- and ψ_2^- are finite on all $\mathcal{F}$-sets. To see this, let $F_1 \in \mathcal{F}$ be such that $\psi_1(F) > -1$ for all $F \in \mathcal{F}$ containing F_1. Choosing an arbitrary $F \in \mathcal{F}$, there is an $\mathcal{F}$-set, $F' \supseteq (F \cup F_1)$, hence $\psi(F') > -1$, implying $\psi_1^-(F)$ finite.

a similar argument works

~~Similarly~~ for ψ_2. Since $\psi_1^- + \psi_2^- \geq (\psi_1 + \psi_2)^-$, it follows that the latter is also finite on all $\check{F}$-sets. Hence $\psi_1 + \psi_2$ satisfies (2).

As for (3), we start with the equality

$$(\psi_1 + \psi_2)^+ + \psi_1^- + \psi_2^- = (\psi_1 + \psi_2)^- + \psi_1^+ + \psi_2^+, \qquad (3.4.4)\ \cancel{(4)}$$

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which follows from the equivalence criterion. For $\underline{F} \in \check{F}$, all the lower variations in (4) are finite, hence we may transpose them and combine them with the upper variations to obtain

$$(\psi_1 + \psi_2)(\underline{F}) = \psi_1(\underline{F}) + \psi_2(\underline{F}). \qquad (3.4.5)\ \cancel{(5)}$$

Now, for given $\varepsilon > 0$, choose $\underline{F}_1, \underline{F}_2 \in \check{F}$ so that $\psi_1(\underline{F}) > -\varepsilon/2$ if $\underline{F} \supseteq \underline{F}_1$, and $\psi_2(\underline{F}) > -\varepsilon/2$ if $\underline{F} \supseteq \underline{F}_2$. There is an $\check{F}$-set $\underline{F}_\varepsilon \supseteq (\underline{F}_1 \cup \underline{F}_2)$, and for any $\check{F}$-set $\underline{F}$ containing $\underline{F}_\varepsilon$ we have, from (5),

$$(\psi_1 + \psi_2)(\underline{F}) > -\varepsilon/2 - \varepsilon/2 = -\varepsilon,$$

since $\underline{F} \supseteq \underline{F}_1$ and $\underline{F} \supseteq \underline{F}_2$. Hence $\psi_1 + \psi_2$ satisfies (3). ~~|||~~ $\square$

3rd B

Definition: Let $\check{F}$ be an extension class on measurable space $(\underline{A}, \Sigma)$. The <u>extended order determined by $\check{F}$</u>, $>_{\check{F}}$, on the space of pseudomeasures, is the vector partial order whose positive cone is

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break at word

$$\{\psi \mid \psi \text{ satisfies (2) and (3)}\}.$$

The intuitive notions underlying this construction are as follows. The extension class $\mathcal{F}$ determines a generalized sort of convergence toward the universe set $\underline{A}$ via successively larger sets $\underline{F} \in \mathcal{F}$. Condition (3) then states, roughly, that ψ^+ catches up to ψ^- as $\mathcal{F}$-sets get larger.

Our next result is the crucial property of extended order.

Theorem: For any extension class $\mathcal{F}$ on a measurable space $(\underline{A},\Sigma)$, $\succ_{\mathcal{F}}$ is an extension of standard order, $\succ$.

Proof: First we note that for any extension class $\mathcal{F}$, there is a sequence $(\underline{F}_n)$, $\underline{n} = 1, 2, \ldots$, (finite or infinite) of $\mathcal{F}$-sets such that $\underline{F}_1 \subseteq \underline{F}_2 \subseteq \ldots$, and whose union is A. To see this, let $\{\underline{F}_1', \underline{F}_2', \ldots\}$ be a countable collection of $\mathcal{F}$-sets whose union is $\underline{A}$. Then, successively, these are $\mathcal{F}$-sets $\underline{F}_1 \supseteq \underline{F}_1'$, $\underline{F}_2 \supseteq (\underline{F}_1 \cup \underline{F}_2')$, $\underline{F}_3 \supseteq (\underline{F}_2 \cup \underline{F}_3')$, etc., and these unprimed $\underline{F}_n$'s satisfy the stated conditions.

Now let $\psi \succ 0$ (standard order), so that $\psi^+(\underline{A}) > \psi^-(\underline{A})$. ψ is a signed measure, so its restricted domain is all of Σ. Thus ψ satisfies (2). If $\psi^+(\underline{A}) = \infty$, the sequence $\psi^+(\underline{F}_n)$ surpasses any finite number as $\underline{n} \to \infty$. Hence, for $\underline{N}$ sufficiently large, $\psi^+(\underline{F}_N) \geq \psi^-(A) + 1$, since $\psi^-(\underline{A})$ is finite. If $\psi^+(\underline{A})$ is finite, then

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$$\psi^+(\underline{A}) > \tfrac{1}{2}[\psi^+(\underline{A}) + \psi^-(\underline{A})] > \psi^-(\underline{A}). \qquad (3.4.6)\ \cancel{(6)}$$

The sequence $\psi^+(\underline{F}_n)$ approaches $\psi^+(\underline{A})$ as $\underline{n} \to \infty$. Hence, for N sufficiently large, $\psi^+(\underline{F}_N)$ surpasses the middle term in (6).

Now let $\underline{F}$ be an F-set containing $\underline{F}_N$. We get

$$\psi(\underline{F}) = \psi^+(\underline{F}) - \psi^-(\underline{F}) \geq \psi^+(\underline{F}_N) - \psi^-(\underline{A}) \geq \underline{b}, \qquad \text{(3.4.7)}$$

where $\underline{b}$ is a fixed positive number not depending on $\underline{F}$. This implies that $\psi >_F 0$. Also $(-\psi) >_F 0$ is false, since any F-set has a larger F-set, $\underline{F}$, satisfying (7), so that $(-\psi)(\underline{F}) \leq -\underline{b}$. Thus $\psi >_F 0$.

For any pair of pseudomeasures, $\psi_1 > \psi_2$ implies $(\psi_1 - \psi_2) > 0$, so that $(\psi_1 - \psi_2) >_F 0$ which implies $\psi_1 >_F \psi_2$. This establishes the desired implication for strict inequality.

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18 Next, let $\psi \sim 0$, so that $\psi^+(\underline{A}) = \psi^-(\underline{A}) < \infty$. Again (2) is satisfied. Taking an F-sequence $\underline{F}_1 \subseteq \underline{F}_2 \subseteq \ldots$ whose union is $\underline{A}$, both $\psi^+(\underline{F}_n)$ and $\psi^-(\underline{F}_n)$ increase to their common limit $\psi^+(\underline{A})$. Hence for any $\varepsilon > 0$ there is an $\underline{F}_n$ such that, for any F-set $\underline{F}$ containing $\underline{F}_n$, both $\psi^+(\underline{F})$ and $\psi^-(\underline{F})$ lie in the interval

$$[\psi^+(\underline{A}) - \varepsilon/2,\ \psi^+(\underline{A})].$$

Hence $|\psi(\underline{F})| = |\psi^+(\underline{F}) - \psi^-(\underline{F})| < \varepsilon$, which implies $\psi \sim_F 0$.

Finally, $\psi_1 \sim \psi_2$ implies $(\psi_1 - \psi_2) \sim 0$, hence $(\psi_1 - \psi_2) \sim_F 0$, so that $\psi_1 \sim_F \psi_2$. ~~This completes the proof.~~ □

24 This is a very comforting theorem, but it does not guarantee that any particular extended order $>_F$ is a proper extension of $>$, i.e. ~~– that is~~, that some pairs of pseudomeasures not comparable under $>$ become comparable under $>_F$. And in fact there is one case when $>_F$ is definitely not a proper extension;

~~this is~~ when the universe set A belongs to F. For in this case, by (2) the only possible pseudomeasures comparable to 0 are signed measures, which are already all comparable under standard order; hence $\succ_F$ and $\succ$ coincide. We can only hope for a proper extension when A does not belong to F.

We ~~shall~~ now show that the "overtaking" criterion developed in recent years[~~19~~ 20] is just a special extended order. Let p_1 and p_2 be alternative development policies leading to "payoff streams" $\int_0^{\infty} f(p_i,t)\,dt$ $(i = 1, 2$, respectively). Then policy p_1 is said to catch up to policy p_2 iff

$$\liminf_{t^o \to \infty} \int_0^{t^o} [f(p_1,t) - f(p_2,t)]\,dt \geq 0. \qquad \text{(3.4.8)} \; \text{~~(8)~~}$$

That is, for all $\varepsilon > 0$ there is a $t^o(\varepsilon)$ such that, for all $t \geq t^o(\varepsilon)$, the integral in (8) exceeds $-\varepsilon$.[~~20~~ 21] But this is precisely the extended ordering $\succ_F$ that arises from the extension class F consisting of all closed intervals $[0,a]$, $a \in A$ (where universe set A here is the non-negative real half-line).

~~Actually~~ this account is oversimplified in one respect. There are ~~in fact~~ a number of minor variants of this criterion, some found in the literature, others of which can be devised. However, most of these others turn out not to be extensions of standard order. Now standard ordering is intuitively much more compelling than any overtaking variant. Hence if some pairs of pseudomeasures are given one order by $\succ$, and a different order by some other criterion, this constitutes grounds for dropping the other criterion as counter-intuitive. We ~~shall~~ therefore do not bother with any variants of the overtaking criterion other than "catching-up-to".

This criterion <u>is</u> a proper extension of standard order; e.g., ~~For example~~ the pair of measures μ, ν in (1) are not comparable under $\succ$. But — taking the ordered partition ($\underline{A}_1$, $\underline{A}_2$, ...) to represent successive intervals on the half-line $[0,\infty)$ — they <u>are</u> comparable under "catching-up", and in fact $\mu \succ_{\check{F}} \nu$.

The "catching-up" criterion appears to be somewhat specialized, and it is not immediately clear how to generalize it to spaces other than the real half-line. We now show, however, that it can be re-interpreted in a way ~~which~~ that generalizes to <u>any</u> metric space.

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<u>Theorem</u>: Let $\underline{A}$ be the non-negative real half-line; let $\check{F}$ be the extension class of all closed intervals $[0,\underline{a}]$, $0 \leq \underline{a} < \infty$; and let $\check{G}$ be the extension class of all closed intervals $[\underline{a}_1, \underline{a}_2]$, $0 \leq \underline{a}_1 \leq \underline{a}_2 < \infty$. Then the extended orderings determined by these, $\succ_{\check{F}}$ and $\succ_{\check{G}}$, are identical.

<u>Proof</u>: Special case of next theorem. |||

D

<u>Theorem</u>: Let $\check{F}$ and $\check{G}$ be two extension classes on measurable space $(\underline{A},\Sigma)$, satisfying: (i) $\check{F} \subseteq \check{G}$, (ii) for all $\underline{G} \in \check{G}$ there is an $\underline{F} \in \check{F}$ with $\underline{F} \supseteq \underline{G}$, and (iii) any $\check{G}$-set containing an $\check{F}$-set is itself an $\check{F}$-set. Then $\succ_{\check{F}}$ and $\succ_{\check{G}}$ are identical.

A (iii) align (i) right

We use an obvious notation for $\mathcal{F}$-sets and $\mathcal{G}$-sets.

Proof: Let $\psi >_{\mathcal{F}} 0$, so that $\mathcal{F} \subseteq \Sigma_\psi$; and $F \supseteq F_\varepsilon$ implies $\psi(F) > -\varepsilon$, all $\varepsilon > 0$. For any G, there is an $F \supseteq G$; since $F \in \Sigma_\psi$, we have $G \in \Sigma_\psi$; thus $\mathcal{G} \subseteq \Sigma_\psi$. Also, F_ε is itself a $\mathcal{G}$-set; and if $G \supseteq F_\varepsilon$, then $G \in \mathcal{F}$, so that $\psi(G) > -\varepsilon$. This proves that $\psi >_{\mathcal{G}} 0$.

Conversely, let $\psi >_{\mathcal{G}} 0$. Since $\mathcal{G} \subseteq \Sigma_\psi$, $\mathcal{F} \subseteq \Sigma_\psi$. $G \supseteq G_\varepsilon$ implies $\psi(G) > -\varepsilon$. For F_ε we choose any $\mathcal{F}$-set containing G_ε. Then if $F \supseteq F_\varepsilon$, $F \supseteq G_\varepsilon$; hence $\psi(F) > -\varepsilon$, since F is also a $\mathcal{G}$-set. This proves that $\psi >_{\mathcal{F}} 0$. ∎

Thus "catching-up" is also the order determined by the class of closed bounded intervals on the real half-line. This suggests that for any metric space (in which all closed discs are measurable) a natural generalization of the "catching-up" criterion would be to use the extended order determined by the class of all closed discs. We shall actually use this procedure for the Löschian problem, in which the universe set A is the plane. (One could also use open discs and open intervals throughout instead of closed).[21]

We ~~Let us~~ now return to the study of extended orders in general.

Theorem: Let (A, Σ) be a measurable space, and ψ_1, ψ_2 two pseudo-measures ~~which are~~ not comparable under standard order. Then ~~there~~ exists an extension class $\mathcal{F}$ such that $\psi_1 >_{\mathcal{F}} \psi_2$.

<u>Proof</u>: Let $\psi = \psi_1 - \psi_2$. We must show that $\psi >_{\mathcal{F}} 0$ for some extension class $\mathcal{F}$. Let (P,N) be a Hahn decomposition for ψ. Since ψ is not comparable to 0 under standard order, ψ^+ and ψ^- are both infinite; in particular, $\psi^+(P) = \infty$.

Let $\{N_1, N_2, \ldots\}$ be a countable measurable partition of N, such that $\psi^-(N_i)$ is finite, all i. For $\mathcal{F}$ we take the class $\{P \cup N_1, P \cup N_1 \cup N_2, \ldots\}$. ~~It is clear that~~ this is clearly an extension class. Also, for each set $F \in \mathcal{F}$, $\psi^+(F) = \infty$, and $\psi^-(F) < \infty$, which implies $\psi >_{\mathcal{F}} 0$. $\Box$

By symmetry there is another extension class $\mathcal{G}$ giving the opposite inequality: $\psi_2 >_{\mathcal{G}} \psi_1$. This underscores the great diversity among the possible extended orders, and the need for some "rational" selection among them (in the occasional cases in which standard order does not suffice).

Any pseudomeasure not already comparable to 0 can be made comparable under the appropriate extension class $\mathcal{F}$. In general, $\mathcal{F}$ depends on the pseudomeasure. Can one make the stronger claim that there is an $\mathcal{F}$ ~~which~~ that simultaneously makes all pseudomeasures comparable to 0 (hence to each other)? Our last theorem shows that the answer is no, except in a trivial case.

<u>Theorem</u>: Let (A,Σ) be a measurable space. No extended order on the set of pseudomeasures is complete, unless Σ is a finite ~~sigma~~ σ-field.

Proof: If Σ is finite, ~~then~~ standard order is already complete, and all extended orders coincide with $\succ$.

Conversely, suppose there is an extension class $\mathcal{F}$ such that $\succ_{\mathcal{F}}$ is complete. Then $\mathcal{F} \subseteq \Sigma_\psi$ for all pseudomeasures ψ, by (2). Suppose there were a set $F \in \mathcal{F}$ ~~which~~ contain~~ed~~ing an infinite number of measurable sets. The proof that standard order is incomplete if Σ is infinite shows how to construct a pseudomeasure ψ such that $\psi^+(F) = \psi^-(F) = \infty$. Then F would not belong to the restricted domain of ψ. This contradiction shows that each $\mathcal{F}$-set contains at most a finite number of measurable sets.

Let $F_1 \subseteq F_2 \subseteq \ldots$ be an increasing sequence of $\mathcal{F}$-sets whose union is A. We may assume that $F_1 \neq \emptyset$, $F_{n+1} \backslash F_n \neq \emptyset$ for all $n = 1, 2, \ldots$. We ~~shall~~ assume this sequence is infinite, and reach a contradiction. Let ψ be a pseudomeasure such that $\psi(F_1) = 1$, $\psi(F_{n+1} \backslash F_n) = 2$ if n is even, and $\psi(F_{n+1} \backslash F_n) = -2$ if n is odd. (Since the number of measurable sets in each of these sets is finite, it is trivial to construct such a ψ.) Then $\psi(F_n) = 1$ if n is odd, and $\psi(F_n) = -1$ if n is even.

Now let F be any $\mathcal{F}$-set. The measurable sets contained in F are generated by a finite partition $\mathcal{G}$ of F. Each $G \in \mathcal{G}$ is contained in some F_n of the sequence $F_1 \subseteq F_2 \subseteq \ldots$; hence F is contained in the union of these, which is another F_n. Also, $\psi(F_n)$ and $\psi(F_{n+1})$ take on the values +1, -1 in some order. Hence for any $F \in \mathcal{F}$, there are $\mathcal{F}$-sets F', F'', each containing F, with $\psi(F') = 1$,

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$\psi(\underline{F}'') = -1$. This shows that ψ is not comparable to 0 under $\succ_F$, contradicting the assumption of completeness of $\succ_F$.

Thus the sequence $\underline{F}_1 \subseteq \underline{F}_2 \subseteq \ldots$ is finite, so Σ is finite. |||

Note that for the case in which some $\succ_F$ is complete, standard order is itself already complete. Thus the situation is this: If standard order is not complete, ~~then,~~ while some of the gaps can be filled by using one or another extended order, it is impossible to fill all of the gaps. We close on this slightly pessimistic note.

B

Conclusion

Standard order on the vector space of pseudomeasures over measurable space $(\underline{A},\Sigma)$ has great intuitive appeal as a representation of preferences. It appears to carry one satisfactorily through the great bulk of problems ~~which~~ that arise. (Standard order suffices for 99% of this book; only in the ~~very~~ last subsection of the ~~very~~ last section of the last chapter do we go beyond it).

When the incompleteness of standard order causes trouble, ~~one~~ we can use an extended ordering to fill in some of the gaps. The problem here is to choose the appropriate extension. The most appealing choice ~~that has been~~ suggested is the "catching-up" or "overtaking" criterion. This has been applied to some special cases, and the generalization suggested here is to the extended order determined by the closed discs (or the open discs) of a metric space. The intuitive idea here is that "nearby" positive and negative masses may be cancelled.

In all interesting cases even extended ordering remains incomplete.

FOOTNOTES - CHAPTER 3

[1]C. Carathéodory defines a similar operation in his Algebraic Theory of Measure and Integration, F. E. J. Linton, translator, P. Finsler, A. Rosenthal, R. Steuerwald, editors (Chelsea, New York, 1969), pages 299-304, in the context of "regular outer measures" on a "sigma-ring of somas". But he then takes the difference of the two variations (on the somas where this is well-defined) and thereby misses the following theory, which depends on retaining λ^+, λ^- as separate entities.

[2]For discussion of infima and suprema of measures, see N. Dunford and J. T. Schwartz, Linear Operators, vol. 1, pages 162-163.

[3]For the concepts involved see N. Jacobson, Lectures in Abstract Algebra, vol. I (Van Nostrand, Princeton, 1951), pages 162-167.

[4]A good general reference is R. Duncan Luce and H. Raiffa, Games and Decisions (Wiley, New York, 1957).

[5]The material in this sub section and the next is well known, but terminology is not completely standardized, and we have selected those aspects ~~which~~ that are relevant for our particular purposes.

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[6]The customary mathematical term for this concept is "maximal".

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[7]A point is said to be Pareto efficient for the family of partial orders ($\succ_i$), $i \in I$, over H iff it is unsurpassed in their Pareto ordering.

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[8]See, ~~for example~~ e.g., J. L. Kelley, I. Namioka, et al, *Linear Topological Spaces* (Van Nostrand, Princeton, 1963), p. 16.

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[9]This relation may be written $\mu > \nu$. This notation is consistent with the corresponding pseudomeasure relation $(\mu,0) > (\nu,0)$. Note that $\mu > \nu$ does not mean that $\mu(E) > \nu(E)$ for all $E \in \Sigma$. In fact, the latter condition never holds, since all measures are zero ~~for~~ at $E = \emptyset$.

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[10]This ambiguous usage should cause no confusion. We ~~shall~~ also consider below some "non-faithful" representations, in which either the "if" or the "only if" of (11) is relaxed.

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11. [11]F. P. Ramsey, "A Mathematical Theory of Saving," Economic Journal, 38:543-559, ~~December,~~ 1928. Reprinted in Readings in Welfare Economics, K. J. Arrow and T. Scitovsky, edi~~tors~~ (Irwin, Homewood, Ill., 1969), ~~pages~~ pp. 619-633.

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12. [12]In this probabilistic context, f is usually called a random variable, with expectation c.

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13. [13]in J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior (Princeton Univ~~ersity~~ Press, Princeton, N.J., 1947), 2nd ed~~ition, 1947),~~ ~~pages~~ pp. 617-628. The axiomatic discussion was flawed, one of the basic axioms being concealed implicitly in an operation, as ~~was~~ noted by E. Malinvaud, (repr~~inted~~ on p~~age~~ 271 of ~~the~~ Readings, mentioned in ~~the next footnote~~ fn. 14). The idea of maximizing expected utility originates with Daniel Bernoulli, 1738.

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14. [14]In so-called "mixture spaces" the utility function need not be bounded, but here it is not expressed in the form of an integral. See I. N. Herstein and J. Milnor, "An Axiomatic Approach to Measurable Utility", Econometrica, 21:291-297, ~~April, 1953,~~ (Apr. 1953) reprinted in Readings in Mathematical Economics, vol. I, P. Newman, edi~~tor~~ (Johns Hopkins Press, Baltimore, 1968), ~~pages~~ pp. 264-270. In our lemma below we use a theorem of Fishburn ~~which~~ that is very similar to the Herstein-Milnor result.

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15. Actually, the following theorems and proofs generalize easily to the class of all discrete probabilities (i.e., concentrated on a countable set) over a general measurable space; but the resulting utility function ~~is~~ may not be measurable.

16 ~~15~~ R. J. Aumann, "Utility Theory Without the Completeness Axiom," *Econometrica*, 30:445–462, 1962.

17 ~~16~~ P. C. Fishburn, *Utility Theory for Decision Making* (Wiley, New York, 1970), p. 107.

18 ~~17~~ This result is well known. See, for example, Fishburn, *Utility Theory* ~~for Decision Making~~, p. 107.

19 ~~18~~ K. J. Arrow, *Essays in the Theory of Risk-Bearing* (Markham, Chicago, 1971), ~~pages~~ pp. 48–49, 63–65.

20 ~~19~~ See C. C. von Weizsäcker, "Existence of Optimal Programmes of Accumulation for an Infinite Time Horizon," *Review of Economic Studies*, 32:85–104, (April 1965); *Review of Economic Studies*, ~~vol.~~ 34, (January 1967) (entire issue is on Optimal Infinite Programmes). For comparison with the earlier and distinct approach of Ramsey see ~~above, page~~ discussion at (3.16)–(3.18).

21 ~~20~~ D. Gale, "On Optimal Development in a Multi-Sector Economy," *Review of Economic Studies*, 34:1–18, (January 1967), pp. 2–3.

22 ~~21~~ Concerning closed versus open discs, we can prove the following result: The pseudomeasure orderings determined by the following three extension classes — the class of open discs; the class of closed discs; and the class of all discs, open and closed — are identical, provided at least one of the following conditions is satisfied: either (i) every closed disc is compact, or (ii) any two points belong to a set isometric to the real line. (Both these conditions are satisfied by the Euclidean and city-block metrics, for example. For "compactness" see Section 7.4; for "isometry" see Section 2.7).