Conditional Equilibrium Solution to a One-Dimensional Kalman Filter

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Let $x_t = ax_{t-1} + u_t$, $y_t = x_t + v_t$, $t = 0, \pm 1, \pm 2, \ldots$,

where all r.v.'s are real valued, $u_t \sim N(o,u)$, $v_t \sim N(o,v)$, the u_t's and the v_t's are all independent of each other, and x_s is independent of u_t, v_t for all $s<t$. $0<|a|<1$.

Problem: Find the joint distribution of the x_t's, for $t=0, -1, -2, \ldots$, given the y_t's, $t=0, -1, -2, \ldots$

The joint distribution being Gaussian, it is determined by two infinite matrices: the covariance matrix W, given by $w_{ij} = \mathrm{Cov}(x_{-i}, x_{-j})$, $i,j = 0,1,2,\ldots$, and the regression matrix A, given by (a_{ij}), where

$$E(x_{-i}|y) = \Sigma_{j=0}^{\infty} a_{ij} y_{-j}, \; i = 0,1,2,\ldots$$

We will derive explicit formulas for w_{ij} and a_{ij} in terms of the three basic parameters u,v and a.

The basic procedure is the following:

1 Given $y = (y_0, y_{-1}, y_{-2}, \ldots)$, extend W to the two extra variables x_1 and y_1.

2 Update by conditioning on y_1. This reproduces the original situation of the joint distribution of the x_t, $t \leq T$, conditional on the y_t, $t \leq T$, except that now $T=1$ instead of $T=0$.

3 The W and A matrices must then reappear shifted in time. This yields a system of recursive equations which can be solved explicitly.

(The solution for W is straightforward. The solution for A is more involved. It turns out that

$$A=W/v \tag{1}$$

There is a direct heuristic argument for (1)).

(1) Write $w = w_{00} = \mathrm{Var}(x_0)$. (Everything is conditional on $y = y_0, y_{-1}, \ldots$)

$$\mathrm{Var}(x_1) = \mathrm{Var}(ax_0 + u_1) = a^2w + u.$$

$$\mathrm{Cov}(x_0, x_1) = a\,\mathrm{Var}(x_0) = aw.$$

$$\mathrm{Cov}(x_{-i}, x_1) = \mathrm{Cov}(x_{-i}, x_0)\, \mathrm{Cov}(x_0, x_1)/\mathrm{Var}(x_0) = aw_{io},$$

since $\Pr(x_1 | x_0, x_{-i}) = \Pr(x_1 | x_0)$.

$$\mathrm{Cov}(x_1, y_1) = \mathrm{Var}(x_1) = a^2w + u.$$

$$\mathrm{Cov}(x_{-i}, y_1) = \mathrm{Cov}(x_{-i}, x_1)\, \mathrm{Cov}(x_1, y_1)/\mathrm{Var}(x_1) = aw_{io},$$

since $\Pr(y_1 | x_1, x_{-i}) = \Pr(y_1 | x_1)$, $i=0,1,2,\ldots$

$$\mathrm{Var}(y_1) = \mathrm{Var}(x_1 + v_1) = a^2w + u + v.$$

Thus the augmented W matrix looks like Figure 1. This completes Step 1.

$$\begin{array}{c|c|c|c|c|c} & y_1 & x_1 & x_0 & x_{-1} & \\ \hline y_1 & a^2w+u+v & a^2w+u & aw & aw_{01} & \cdots \\ \hline x_1 & a^2w+u & a^2w+u & aw & aw_{01} & \cdots \\ \hline x_0 & aw & aw & w & w_{01} & \cdots \\ \hline x_{-1} & aw_{10} & aw_{10} & w_{10} & w_{11} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Figure 1

(2) Condition on y_1. Writing Figure 1 as Figure 2, where $\tilde{W}$ is W bordered by the x_1 row and column, the conditioning yields

$$\tilde{W} - \beta\alpha^{-1}\beta' = \tilde{W} - \beta\beta'/(a^2w + u + v) = W^0,$$

since α is (1,1).

$$\begin{array}{|c|c|} \hline \alpha & \beta' \\ \hline \beta & \tilde{W} \\ \hline \end{array}$$

Figure 2

W^0 is the covariance matrix of $x_1, x_0, x_{-1}, \ldots$, conditional on $y^0 = y_1, y_0, y_{-1}, \ldots$. All the following are conditional on y^0.

$$\mathrm{Var}(x_1) = a^2w+u - \frac{(a^2w+u)^2}{a^2w+u+v} = \frac{v(a^2w+u)}{a^2w+u+v}$$

But this must equal w, by recursion! Thus $w(=w_{00})$ satisfies the quadratic equation

$$w(a^2w+u+v) = v(a^2w+u) \qquad (2)$$

(2) has a unique positive root (it is always less than $\min\left[v, \dfrac{u}{1-a^2}\right]$)

Define the number θ by:

$$\theta = \frac{av}{a^2w+u+v}\left[= \frac{aw}{a^2w+u} = a\left(\frac{v-w}{v}\right)\right] \qquad (3)$$

(The equations in (3) follow easily from (2), and imply $0<\theta/a<1, 0<\theta a<1$, hence $|\theta|<1$.

$$\mathrm{Cov}(x_{-i},x_1) = aw_{io} - \frac{aw_{io}(a^2w+u)}{a^2w+u+v} = \frac{aw_{io}v}{a^2w+u+v} = \theta w_{io}, \text{ by (3)}$$

But this must equal $w_{i+1,0}$, by recursion!

It follows that

$$w_{io} = \theta^i w \tag{4}$$

Finally, for i,j = 0, -1, -2, ...,

$$\mathrm{Cov}(x_{-i},x_{-j}) = w_{ij} - \frac{aw_{io}aw_{jo}}{a^2w+u+v} = w_{ij} - \frac{aw^2\theta^{i+j+1}}{v}$$

by (3) and (4).

But by recursion, $\mathrm{Cov}(x_{-i}, x_{-j})$ must be $w_{i+1,j+1}$!

Thus

$$w_{i+1,j+1} = w_{ij} - \frac{aw^2\theta^{i+j+1}}{v} \tag{5}$$

(5) is a system of difference equations, which, with initial conditions given by (2) and (4), have the unique solution

$$\frac{w_{ij}}{w} = \frac{(1-a\theta)\theta^{|i-j|}+(a-\theta)\theta^{i+j+1}}{1-\theta^2} \tag{6}$$

This completes the solution for the covariance matrix W. For the regression matrix A, let $x = (x_1, x_0, x_{-1}, x_{-2}, ...)$ and, as above, $y = y_0, y_{-1}, ...$ and $y^0 = y_1, y_0, y_{-1}, ...$ Then

$$E(x|y^0) = E(x|y) + \beta\alpha^{-1}(y_1 - E(y_1|y)) \tag{7}$$

(see Figure 2) To start, note that

$$E(y_1|y) = E(x_1|y) = aE(x_0|y) = a\Sigma_{j=0}^{\infty}a_{0j}y_{-j} \tag{8}$$

Then, from (7) and (8),

$$E(x_1|y^0) = a\Sigma_{j=0}^{\infty}a_{oj}y_{-j} + \frac{a^2w+u}{a^2w+u+v}(y_1 - a\Sigma_{j=0}^{\infty}a_{0j}y_{-j}) \tag{9}$$

$$= \frac{w}{v}y_1 + \theta\Sigma_{j=0}^{\infty}a_{0j}y_{-j}$$

by (2) and (3). But, by recursion, we must have

$$E(x_1|y^0) = a_{00}y_1 + \Sigma_{j=0}^{\infty}a_{0,j+1}y_{-j} \tag{10}$$

Equating coefficients of y_1 in (9) and (10) yields

$$a_{00} = \frac{w}{v} \tag{11}$$

Equating coefficients of y_{-j} in (9) and (10) yields $a_{0,j+1} = \theta a_{0j}$
so that

$$a_{0j} = \frac{\theta^j w}{v} = \frac{w_{oj}}{v} \tag{12}$$

from (11) and (4).

Similarly, for i = 0, 1, 2, ...

$$E(x_{-i}|y^0) = E(x_{-i}|y) + \frac{aw_{i0}}{a^2w+u+v}(y_1 - E(y_1|y)) \tag{13}$$

$$= \Sigma_{j=0}^{\infty}a_{ij}y_{-j} + \frac{\theta^{i+1}w}{v}(y_1 - a\Sigma_{j=0}^{\infty}a_{0j}y_{-j})$$

from (7), (3), (4), and (8). But, by recursion,

$$E(x_{-i}|y^0) = a_{i+1,0}y_1 + \Sigma_{j=0}^{\infty}a_{i+1,j+1}y_{-j} \tag{14}$$

Equating coefficients of y_1 in (13) and (14) yields

$$a_{i0} = \frac{\theta^i w}{v} = \frac{w_{i0}}{v} \tag{15}$$

by (4), for $i = 1, 2, \ldots$

Equating coefficients of y_{-j} in (13) and (14) yields

$$a_{i+1,j+1} = a_{ij} - \frac{a\theta^{i+1} w a_{0j}}{v} = a_{ij} - \frac{a\theta^{i+j+1} w^2}{v^2} \tag{16}$$

from (12). This difference equation system, together with initial conditions (12) and (15), has the unique solution $a_{ij} = w_{ij}/v$, $i,j = 0,1,\ldots$, as is clear on dividing equations (5) by v. Thus

$$A = W/v \tag{17}$$

This, with (6) (and (2) and (3)) gives the complete solution.

A <u>heuristic derivation of equation</u> (17)

Let $x = (x_0, x_{-1}, x_{-2}, \ldots)'$, $y = (y_0, y_{-1}, y_{-2}, \ldots)'$. We find the <u>unconditional</u> joint covariance matrix over (y', x').

Step 1: Since $x_t = ax_{t-1} + u_t, |a|<1$, the equilibrium covariance matrix on X is easily seen to be

$$\mathrm{Cov}(x_{-i}, x_{-j}) = \frac{u}{1-a^2} a^{|i-j|}, \; i,j = 0,1,2,\ldots$$

Call this infinite matrix Ω.

Step 2: Since $y_t = x_t + v_t$, it follows that

$$\mathrm{Cov}(x_{-i}, y_{-i}) = \mathrm{Var}(x_{-i}) = u/(1-a^2)$$

Also

$$\mathrm{Cov}(x_{-i}, y_{-j}) = \mathrm{Cov}(x_{-i}, x_{-j})\mathrm{Cov}(x_{-j}, y_{-j})/\mathrm{Var}(x_{-j})$$

$$= \frac{u}{1-a^2}\, a^{|i-j|}$$

since $\Pr(y_{-j}|x_{-j}, x_{-i}) = \Pr(y_{-j}|x_{-j})$

This shows that the covariance matrix between x and y is also Ω.

Step 3: $\mathrm{Var}(y_{-i}) = \mathrm{Var}(x_{-i}) + v = \dfrac{u}{1-a^2} + v$

For $i \neq j$, $\mathrm{Cov}(y_{-i}, y_{-j}) = \mathrm{Cov}(y_{-i}, x_{-i})\ \mathrm{Cov}(x_{-i}, y_{-j})/\mathrm{Var}(x_{-i}) = \dfrac{u}{1-a^2}\, a^{|i-j|}$

since $\Pr(y_{-i}|x_{-i}, y_{-j}) = \Pr(y_{-i}|x_{-i})$

Thus the covariance matrix of the y_{-i}'s is $\Omega + vI$ (I the infinite unit matrix)

Thus the overall covariance matrix is as in Figure 3.

$$\begin{array}{c|c|c|} & y' & x' \\ \hline y & \Omega + vI & \Omega \\ \hline x & \Omega & \Omega \\ \hline \end{array}$$

Figure 3

Then, assuming the rules for finite conditioning carry over to the infinite case, we get for W, the conditional covariance of x conditioned on y,

$$W = \Omega - \Omega(\Omega + vI)^{-1}\Omega \tag{18}$$

For A, the regression matrix of x on y, we get

$$A = \Omega(\Omega + vI)^{-1} \tag{19}$$

Then (18) and (19) imply (17), since

$$W = \Omega - A\Omega = A(\Omega + vI) - A\Omega = Av$$

Further comments:

(i) The conditional distribution of <u>all</u> x_t's, $t=0,\pm1,\pm2,\ldots$ given <u>all</u> y_t's, $t=0,\pm1,\pm2,\ldots$, follows from the above by going to the infinite past. Let $i,j \to \infty$ in (6) while $i-j$ is fixed. The "Hankel" part disappears, leaving only the "Toeplitz" part:

$$w_{ij} = \frac{w(1-a\theta)}{1-\theta^2}\,\theta^{|i-j|}$$

for $i,j=0,\pm1,\pm2,\ldots$ is the doubly infinite covariance matrix. The relation $A=W/v$ still obtains.

(In this setup, w is of no interest, but rather $w_\infty = \frac{w(1-a\theta)}{1-\theta^2}$, which is the common residual variance of the x's; by tedious calculation,

$$w_\infty = \frac{uv}{\sqrt{(u+v-a^2v)^2+4a^2uv}}$$

(ii) The formulas above seem to be meaningful for all real <u>a</u>, not just for $|a|<1$.

(iii) The joint distribution of $x_0, x_1, x_2, \ldots$ conditional on $y_0, y_1, y_2, \ldots$ appears to be of just this form again; w_{ij} and a_{ij} are still given by (2), (3), (6) and (17), where now $w_{ij} = \mathrm{Cov}_y(x_i, x_j)$ and

$$E(x_i|y) = \Sigma_{j=0}^{\infty} a_{ij} y_j$$

The basic argument for this is that the joint covariance matrix of $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$ is again as in Figure 3, with the same Ω. (Alternatively, the stationary joint distribution of x_t, y_t, $t=0, \pm1, \pm2, \ldots$ is <u>time reversible</u>).

(iv)

$$\Sigma_{j=0}^{\infty} a^{i-j} a_{ij} = 1, \text{ all } i = 0,1,\ldots \tag{20}$$

Proof: direct computation from (6) and (17)

A statement equivalent to (20) is: $A\alpha = \alpha$, (21)

where $\alpha = (1, a^{-1}, a^{-2}, \ldots)'$

(v) The W (and A) matrices are positive definite

Proof: From (3) recall that $0<\frac{\theta}{a}<1, 0<\theta a<1$, hence $0<|\theta|<1$

The "Toeplitz" matrix $(\theta^{|i-j|})$ is positive definite by $0<|\theta|<1$, and is multiplied by the positive factor $(1-a\theta)/1-\theta^2)$

The "Hankel" matrix (θ^{i+j}) is positive semidefinite, and is multiplied by the positive factor $(a-\theta)\theta/(1-\theta^2)$. W is the sum of these two. QED

(Note this argument works for any real $a \neq 0$).