The Theory of Comparative Advantage for Many Countries and Many Industries

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Introduction

The theory of comparative advantage, illustrated by Ricardo for the case of two countries and two industries, is among the most splendid achievements of economic science. It was extended in the nineteenth century to the case of two countries and any number of industries, and also to the case of two industries and any number of countries (see Viner, pp. 453-67, Haberler, pp.136-40.) But when the number of countries and industries both exceed two, little progress has been made. Thus we are in the anomalous position of having elaborate theories with any number (even a continuous infinity - see Dornbusch, et. al.(1977)) of industries, provided there are just two countries. But at the same time we are unable to answer certain simple questions even for the 3-by-3 case: for example, to classify and enumerate the possible technologies.

This paper will go some distance to extending the theory to the general m-country n-industry case. The literature on the subject is easily summarized. Pioneering contributions were made by McKenzie (1954) and Jones (1961), and since then the subject has remained frozen in time. Their work on the general theory is discussed in detail in Section I.3 below. In addition, McKenzie contributed to the synthesis problem (our terminology) for three industries. This is discussed in Section II.1 below.

I. The Analytic Theory of Comparative Advantage

I.1. Efficiency and Competitive Prices

We make the usual Ricardian assumptions. There is just one factor of production - called "labor" - in each country which cannot move between countries but is freely mobile and divisible across industries within its own country. An industry consists of a single product whose output is proportional to labor input. Products are freely mobile across countries. Formally we have:

$i = 1, \ldots, m.$

$j = 1, \ldots, n.$

$$L_i = \Sigma_{j=1}^{n} L_{ij} \tag{1}$$

$$X_j = \Sigma_{i=1}^{m} X_{ij} \tag{2}$$

$$L_{ij} = a_{ij} X_{ij} \tag{3}$$

all i,j, with $L_{ij}, X_{ij} \geq 0$.

Here L_i is the labor pool in country i, X_j is the world output of product j, L_{ij} is the labor in country i engaged in industry j and X_{ij} is the corresponding output, and a_{ij} is the technical coefficient, the labor required for producing unit output of product j in country i. All quantities must be non-negative real numbers, with the a_{ij} positive. The m-by-n technology matrix A of the a_{ij}'s is given to the economy. (We could just as easily work with the reciprocals $e_{ij} = 1/a_{ij}$, the labor productivity coefficients.)

What allocations arise under a competitive price system? What allocations are efficient? The answers to these questions are closely connected. Consider first the Output Efficiency Problem: Given technology A and positive numbers $L_1, L_2, \ldots, L_m$, Maximize $(X_1, X_2, \ldots, X_n)$ subject to (1), (2), (3).

Here "maximize" is to be interpreted in the vector sense: an allocation is a solution if no X_j can be increased without another being decreased. The set of all solutions $(X_1, \ldots, X_n)$ constitutes the output efficiency frontier, a hypersurface in the non-negative orthant of n-space.

A standard convexity argument now shows that the output efficiency frontier is traced out by the following set of problems: Maximize $P_1X_1 + \ldots + P_nX_n$, subject to the same constraints as above (where $P_1, \ldots, P_n$ are positive numbers). That is, for any $P_1, \ldots, P_n > 0$ a solution to this problem is efficient and, conversely, any efficient $X_1, \ldots, X_n$ is a solution to a problem of this sort for some $P_1, \ldots, P_n > 0$.

Now, these new problems are linear programs, trivial ones whose solutions can be written out explicitly as follows: For country i, calculate the n ratios: $P_1/a_{i1}, P_2/a_{i2}, \ldots, P_n/a_{in}$. Let W_i be the maximum of these ratios. If $P_j/a_{ij} < W_i$, set L_{ij} (and X_{ij}) equal to zero, and distribute L_i among the remaining j's (the ones tied for the largest ratio) in any arbitrary manner.

If the P's are interpreted as prices, this describes exactly the behavior of income-maximizing workers in a competitive environment. For, P_j/a_{ij} is the unit earnings to be made by a worker in country i employed in industry j. Workers move to the industries maximizing this return, the maximum becoming their wage W_i.

The Input Efficiency Problem is: Given technology A and positive numbers $X_1, \ldots, X_n$, Minimize $(L_1, \ldots, L_m)$ subject to (1), (2) and (3). Here "minimize" is to be interpreted in the vector sense again: no L_i can be decreased without another being increased. The set $(L_1, \ldots, L_m)$ of all solutions constitutes the input efficiency frontier, a hypersurface in the non-negative orthant of m-space.

Again a convexity argument shows we may substitute the set of problems: Minimize $W_1L_1 + \ldots + W_mL_m$, with the same feasible set (where $W_1, \ldots, W_m > 0$). These trivial linear programs have the following explicit solutions: For product j, calculate W_1a_{1j}, $W_2a_{2j}, \ldots, W_ma_{mj}$. Let P_j be the minimum of these. If $W_ia_{ij} > P_j$, set X_{ij} (and L_{ij}) to zero, and distribute X_j among the remaining i's (the ones tied for smallest) in any arbitrary manner. Interpreting the W's as wages, this mimics the behavior of buyers seeking the least cost producers, this lowest cost becoming the price P_j.

The Input Efficiency Problem is a bit artificial perhaps. But the point is to obtain structural information concerning patterns and configurations. It turns out that either efficiency frontier gives a complete coding of this information. If $m < n$, it will be easier to obtain this information from the input efficiency frontier, which is in the lower-dimensional space.

(Further, the magnitudes of $X_1, \ldots, X_n$ turn out to be irrelevant for this structure; they need only be positive.)

For example, the 2-country, n-industry case has an input efficiency frontier that is a polygonal arc, convex to the origin, with n edges and n+1 vertices. The output efficiency frontier, on the other hand, is a hyperpolyhedron in n-space which one would rather not think about.

I.2. Viable Patterns

The theory of comparative advantage is concerned with the patterns describing which countries engage in which industries. Formally, a <u>pattern</u> is an (m,n) matrix S of zeros and ones, where $s_{ij} = 1$ if L_{ij} (and X_{ij}) are positive, and $s_{ij} = 0$ if L_{ij} (and X_{ij}) are zero.

As we sweep across the output or input efficiency frontier, or as we vary the P's or W's, a range of different patterns arise, a <u>configuration</u> of patterns. The configurations themselves depend on the technologies A. We are searching for general laws arising at all three levels - within patterns, across patterns in configurations, and across technologies.

Solutions to either the Input or the Output Efficiency Problems satisfy the same price-wage relations: There exist $P_1, \ldots, P_n, W_1, \ldots, W_m > 0$ such that, for all i and j,

$$P_j \leq W_i a_{ij} \tag{4}$$

$$\text{If } s_{ij} = 1, \text{ then } P_j = W_i a_{ij} \tag{5}$$

The inequalities (4) are immediate. As for (5), it merely restates that if $P_j < W_i a_{ij}$ for some (i,j), then L_{ij} and X_{ij} are zero.

In addition, in the (Output,Input) Efficiency Problem there are no all-zero (rows, columns), since the positive quantities of (resources, outputs) must be allocated somewhere. This suggests the following definitions.

Relative to technology A, a pattern S is <u>viable</u> if there exist $P_1, \ldots, P_n, W_1, \ldots, W_m > 0$ satisfying (4) and (5). S is <u>output efficient</u> if it is viable with all row sums positive. S is <u>input efficient</u> if it is viable with all column sums positive. S is <u>efficient</u> if it is both input and output efficient.

(Example: S identically zero is viable, but neither input nor output efficient.)

I.3. The Generalized McKenzie-Jones Principle

In the classical 2-by-2 technology shown, suppose ad < bc, so that England (Portugal) has a comparative advantage in producing cloth (wine) respectively.

The classical result is that a pattern is viable if and only if it does <u>not</u> entail <u>both</u> England producing wine and Portugal cloth.

	Cloth	Wine
England	a	b
Portugal	c	d

Figure 1

For the m-by-n case, viability requires a similar condition for every 2-by-2 submatrix. But (if m, n > 2) even the satisfaction of all these conditions does not guarantee viability. The first to realize this was Lionel McKenzie (unpublished letter referring to McKenzie (1954), where the example arose as an error. The first published result was Jones (1961); see his p. 163, n. 3, for this curious bit of intellectual history).

To get at the extra conditions, take any set of r distinct countries and any set of r distinct industries. For simplicity label both the countries and the industries by the numbers 1, ..., r. A <u>match</u> , σ, is a 1-1 correspondence between these two sets: $\sigma(i)$ is the label of the industry matched with country i. The <u>score</u> of match σ is the product of the technical coefficients: $a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{r,\sigma(r)}$.

There are r! possible matches between these two sets. Consider all their scores. A <u>Jones match</u> is one that attains a minimal score. (There may be ties for minimal score, in which case each is a Jones match). Note that these are determined solely by the technology matrix A, and that each square submatrix determines a Jones match between the countries of its rows and the industries of its columns.

Now consider a pattern S. Let i(1), ..., i(r) be r distinct countries and j(i), ..., j(r) be r distinct industries such that $s_{i(1),j(1)} = 1, \ldots, s_{i(r),j(r)} = 1$. Call the set of pairs {(i(1),j(1)), ..., (i(r),j(r))} a <u>match in S</u>. Suppose S has the property that every match in S is a Jones match. For r = 2 this is exactly the principle of comparative advantage: the correspondence England → wine, Portugal → cloth is <u>not</u> a Jones match while the opposite pairing is. Thus the former must not be a match in S - that is, it must not be both that England produces wine and Portugal cloth. Thus we get all the 2-by-2 conditions mentioned above. But we also get additional conditions for r = 3,4, ... : Every square submatrix of S yields an additional condition that must be met.

To see that these <u>are</u> additional conditions consider the technology of Figure 2, of size n-by-n, where $a_{ii} = 2$ on the diagonal, $a_{ij} = 1$ on the superdiagonal and the corner as shown, and $a_{ij} = M$ elsewhere, where $M > 2^{n-1}$.

2	1	M	•	•	•	M
M	2	1	•			•
•	M	•	•	•		•
•		•	•	•	•	•
•			•	•	•	M
M				•	2	1
1	M	•	•	•	M	2

Figure 2

Consider the pattern S with $s_{ii} = 1$, $s_{ij} = 0$, all $i \neq j$ (the unit matrix). The diagonal match in S is not a Jones match, since $2^n > 1$. But every other match in S <u>is</u> a Jones match: For r < n an (r,r) principal submatrix scores $2^r < M$ for its diagonal match, while all other matches in it score $\geq M$. Thus the violation appears only at the highest level r = n.

In Jones (1961), Ronald Jones considered the special case of n-by-n patterns with a unique match σ, i.e., $s_{i,\sigma(i)} = 1$, all i, and = 0 otherwise. He argued that S is efficient if and only if σ is a Jones match (in our terminology). This latter condition actually implies the apparently stronger condition discussed above, that all matches in S are Jones matches. This in turn follows from the proposition: Any submatch of a Jones match is a Jones match. (The proof is by contradiction: Suppose, e.g., {(1,1), ..., (5,5)} is a Jones match but {(1,1), (2,2), (3,3)} is not. Then $a_{11}a_{22}a_{33} > a_{12}a_{23}a_{31}$ (say). But then $a_{11}a_{22}a_{33}a_{44}a_{55} > a_{12}a_{23}a_{31}a_{44}a_{55}$, so {(1,1), ..., (5,5)} is not Jones after all).

We now generalize these statements to any pattern S of any size m-by-n.

A word about methods of proof. Jones (1961) and McKenzie (1954) are both intellectual tours-de-force, especially the latter. Because so much territory is covered in a few pages, some of the inferences are rather large leaps. In particular, the statement, "... if there is no circuit which can cause the production of a final good to increase, the output vector is efficient" (McKenzie, p. 171), is not at all obvious. (A circuit is a re-allocation as in Figure 3 below). Attempts to justify this inference inevitably lead to linear programming (or linear inequality) type arguments. This explains the appearance of linear programs in this section, as well as in Sections I.4. and I.9. This not only yields clearer proofs than in McKenzie or Jones (Jones implicitly makes the same inference as McKenzie), but (usually) more powerful results.

H. W. Kuhn (pp. 70-75) has also applied linear programming methods to this problem, but only to the special case considered by Jones.

Definition: Given technology A, a pattern S satisfies the generalized McKenzie-Jones principle, or the generalized principle of comparative advantage, if every match in S is a Jones match.

Theorem 1. (The Fundamental Theorem of Comparative Advantage) Pattern S is viable if and only if S satisfies the generalized McKenzie-Jones principle.

Proof. Only if. Let S be viable, and $P_1, \ldots, P_n, W_1, \ldots, W_m > 0$ a price system satisfying (4) and (5). Let σ be a match in S. We may label the set of countries involved and the set of industries involved both by 1, ..., r, and take $\sigma(i) = i$, without loss of generality. Then $P_i = W_i a_{ii}$ since $s_{ii} = 1$ by (5), $i = 1, \ldots, r$. Let τ be any other match between these two sets. Then $P_{\tau(i)} \leq W_i a_{i,\tau(i)}$ by (4), $i = 1, \ldots, r$. Multiplication yields $W_1 \ldots W_r a_{11} \ldots a_{rr} = P_1 \ldots P_r = P_{\tau(1)} \ldots P_{\tau(r)} \leq W_1 \ldots W_r a_{1,\tau(1)} \ldots a_{r,\tau(r)}$.

(The middle equation arises from τ merely rearranging the P's). Cancel the W's:

$$a_{11} \ldots a_{rr} \leq a_{1,\tau(1)} \ldots a_{r,\tau(r)}$$

It follows that σ is a Jones match.

If. Assume S is not viable.

Step 1: Let $t_{ij} = \log a_{ij}$. There do not exist numbers $\pi_1, \ldots, \pi_n$,

$\omega_1, \ldots \omega_m$ satisfying all the following conditions:

$$\pi_j + \omega_i \le t_{ij}, \text{ all } i,j, \text{ with equality if } s_{ij} = 1. \tag{6}$$

For, if such existed, then taking antilogs in (6) would yield $P_1, \ldots, P_n, W_1, \ldots, W_m > 0$ satisfying (4) and (5) and S would be viable — namely, $P_j = \exp \pi_j$, $W_i = \exp(-\omega_i)$.

Step 2: Consider the linear program: Minimize $\Sigma_{i=1}^m \Sigma_{j=1}^n t_{ij} v_{ij}$ subject to

$$\begin{aligned} \Sigma_{j=1}^n v_{ij} &= 0, \text{ all } i, \\ \Sigma_{i=1}^m v_{ij} &= 0, \text{ all } j, \quad \text{and} \\ v_{ij} \ge 0 \text{ for all } i,j &\text{ such that } s_{ij} = 0. \end{aligned} \tag{7}$$

Suppose this program had an optimal solution. Then, by programming theory (Murty, p. 192), the dual would also have an optimal, hence a feasible solution. But the feasibility conditions for the dual are of the form (6), hence have no solution. It follows that program (7) has no optimal solution.

Step 3: There exist v_{ij} which are feasible for (7) and also satisfy $\Sigma_i \Sigma_j t_{ij} v_{ij} < 0$. For, the value 0 is attainable in (7) by taking all $v_{ij} = 0$. Since (7) has no optimal solution, there must be feasible v_{ij} making the objective function negative.

Step 4: Arrange these v_{ij} in an (m,n) matrix V. Since V is not all 0's, there is a "+" entry, then a "-" entry in the same column, then a "+" entry in the same row, etc. (since all row, column sums are 0). Eventually we return to a previously scanned row or column, yielding a cycle of labeled entries as in Figure 3, with each row and column containing either no labels or a single +,- pair. Let $v^1 > 0$ be the smallest absolute value of the labeled entries, and let V^1 be the matrix having entries $+v^1$, $-v^1$ at the +, - labels, and 0 elsewhere. $V - V^1$ still has all row and column sums 0, has at least one more zero entry than V, and does not reverse any V signs. If $V - V^1$ is not all zeros, construct matrix V^2 from it by the same procedure, then V^3 from $V-V^1-V^2$, etc. This yields a decomposition $V = V^1 + V^2 + \ldots + V^q$. ($q \le mn$).

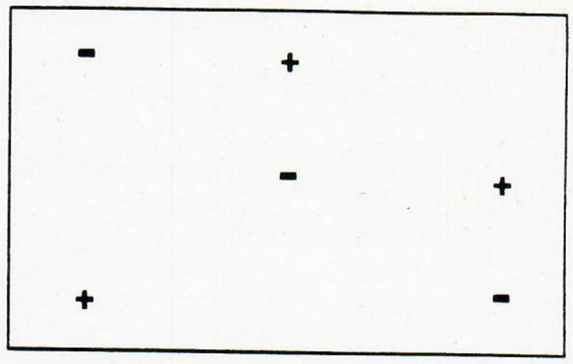

Figure 3

Step 5: Now $0 > \Sigma_i\Sigma_j\, t_{ij}v_{ij} = \Sigma_{k=1}^{q}\Sigma_i\Sigma_j\, t_{ij}v_{ij}^k$, hence

$$0 > \Sigma_i\Sigma_j\, t_{ij}v_{ij}^k \tag{8}$$

for some k = 1, ..., q. By labeling countries and industries appropriately, we may assume that the "-" entries in V^k are at (1,1),..., (r,r), and the "+" entries are at (1,2), (2,3), ..., (r-1,r) and (r,1). After dividing by v^k, (8) then reads

$$t_{11} + t_{22} + \ldots + t_{rr} > t_{12} + t_{23} + \ldots + t_{r-1,r} + t_{r,1} \tag{9}$$

Next, $v_{ii}^k < 0$ implies $v_{ii} < 0$ which in turn implies $s_{ii} = 1$. Hence

$\{(1,1),\ldots, (r,r)\}$ is a match in S. On the other hand, taking antilogs in (9) yields

$$a_{11}a_{22}..a_{rr} > a_{12}..a_{r-1r}a_{r1}$$

so that $\{(1,1),\ldots, (r,r)\}$ is not a Jones match. Thus S does not satisfy McKenzie-Jones. ▒

This result is of basic theoretical importance. It is not a practical method for deciding viability, however, since it requires eximination of every square submatrix. A method that <u>is</u> practical is via the linear program (7). If it has an optimal solution, the dual yields a sustaining price system via the transformation in Step 1.

Examples: (i) Jones' result is a special case of the fundamental theorem, as argued above. (ii) Suppose $a_{ij} = b_i c_j$ for some $b_1,\ldots,b_m$, $c_1,\ldots,c_n > 0$ (A is of rank 1). Then every pattern S is viable, and we may take $P_j = c_j$, $W_i = 1/b_i$ as a price system. Correspondingly, every match is a Jones match.

I.4. The Transportation Problem

The <u>row signature</u> of pattern S of size (m,n) is the m-tuple $(r_1,\ldots,r_m)$, where r_i is the number of "1"s in row i. The <u>column signature</u> is the n-tuple $(c_1,\ldots,c_n)$, where c_j is the number of "1"s in column j. (Note that this also gives the row and column sums of S, respectively, since S is a 0-1 matrix).

We establish a far-reaching connection between viable patterns and the linear programming transportation problem. Let A be a technology of size (m,n) and S a pattern with row, column signatures $(r_1,\ldots,r_m)$, $(c_1,\ldots,c_n)$, respectively (zero values allowed). Let $t_{ij} = \log a_{ij}$, $i = 1,\ldots, m$, $j = 1,\ldots, n$.

Theorem 2. S is viable if and only if (s_{ij}) is optimal for the problem: Minimize $\Sigma_i\Sigma_j\ t_{ij}y_{ij}$, subject to

$$\begin{aligned} \Sigma_j\, y_{ij} &= r_i, \text{ all } i, \\ \Sigma_i\, y_{ij} &= c_j, \text{ all } j, \text{ and} \\ y_{ij} &\geq 0, \text{ all } i,j. \end{aligned} \tag{10}$$

Proof. (s_{ij}) is feasible for this program. A feasible solution is optimal if and only if it satisfies "complementary slackness" with the dual variables, $\omega_1,\ldots, \omega_m, \pi_1,\ldots, \pi_n$ (Murty, p. 199) – that is, if and only if

$$\pi_j + \omega_i \leq t_{ij}, \text{ with equality if } s_{ij} = 1 \tag{11}$$

for all i,j.

Let $P_j = \exp \pi_j$, $W_i = \exp(-\omega_i)$. Taking antilogs in (11) yields (4) and (5), so that "complementary slackness" is equivalent to S being viable. ▒

This gives another characterization of viability, in addition to the generalized McKenzie-Jones condition. (Note that the linear program (7) is a "feasible direction" perturbation of (10)).

Next, for any pattern S define its <u>score</u> as the product of the a_{ij}'s over all (i,j) for which $s_{ij} = 1$ (This generalizes the concept of score for a match).

Theorem 3. If S is viable, then it has minimal score over all other patterns with the same row and column signatures.

Proof. Since S is viable, it optimizes the transportation problem of Theorem 2. Any S' with the same row and column signatures is feasible for this problem, hence $\Sigma_i\Sigma_j\ t_{ij}s_{ij} \leq \Sigma_i\Sigma_j\ t_{ij}s'_{ij}$. Take antilogs. ▒

As a special case, take m = n, and let the row and column signatures be all "1"s. Then we get (half of) Jones' theorem: If a match is viable, it has minimal score! (Jones, p. 166 n.1, recognized the connection between his work and the assignment problem, which is this special case of the transportation problem.)

The converse of Theorem 3 is false: In the 2-by-2 case (Figure 1) let ad < bc, and let S be the (unique) pattern with row and column signatures both (1,2). It is not viable. The point is there may not be a viable pattern with specified row and column signatures, even if $\Sigma_i r_i = \Sigma_j c_j$. (See Theorem 14 below).

I.5. Graphs; Maximal Patterns

It is very useful to introduce some elementary graph theoretic ideas at this point. (See any book on the subject, e.g., Berge (1976). Terminology in this field is not standardized). Associate with any pattern S of size (m,n) a graph consisting of m + n nodes $C_1,\ldots,C_m$, $I_1,\ldots,I_n$ corresponding to the m countries and n industries, and join C_i and I_j with an arc if and only if $s_{ij} = 1$. (The resulting graph is bipartite, since every arc joins a C-type with an I-type node). Two nodes M and N are <u>connected</u> if there is a sequence of nodes $M = x_0$, $x_1, \ldots, x_r = N$ such that each adjacent pair of nodes are joined by an arc. The connectivity relation partitions the set of nodes into connected <u>components</u>. The graph is <u>connected</u> if any two nodes in it are connected. A <u>cycle</u> is a sequence of distinct nodes $x_1, \ldots, x_r$ $(r > 2)$ such that each adjacent pair are joined by an arc, and also x_r, x_1 are so joined. (In a bipartite graph the nodes are alternately C-type and I-type. The arcs in a cycle correspond to "1"s in the S matrix forming a design similar to the labeled entries in Figure 3). A graph is <u>acyclic</u> if it has no cycles. A <u>tree</u> is a connected acyclic graph.

One application of these ideas is to maximal patterns. Pattern S is <u>maximal</u> if it is viable, but such that the change of any s_{ij} from a "0" to a "1" would destroy viability.

Any viable pattern extends to a maximal pattern by successively transforming certain "0"s into "1"s. Conversely, transforming "1"s to "0"s preserves viability. Hence knowing the maximal patterns gives in effect all viable patterns.

Pattern S is <u>strictly sustainable</u> if there is a price system $P_1, \ldots, P_n, W_1, \ldots, W_m > 0$ such that

$$P_j < W_i a_{ij} \quad \text{when } s_{ij} = 0 \qquad (12)$$

$$P_j = W_i a_{ij} \quad \text{when } s_{ij} = 1 \qquad (13)$$

This property is stronger than mere viability since, unlike (4) and (5), it disallows $s_{ij} = 0$ and $P_j = W_i a_{ij}$ holding simultaneously. We may now characterize maximal patterns.

Theorem 4. Under technology A, pattern S is maximal if and only if S has a connected graph and is strictly sustainable.

Proof. (i) "Maximal" <u>implies</u> "connected graph":
Suppose the graph of viable S is not connected. Then there is a partition of countries into sets C', C" and of industries into I', I", such that no arc joins a C' to an I" node, or a C" to an I' node. (See Figure 4).

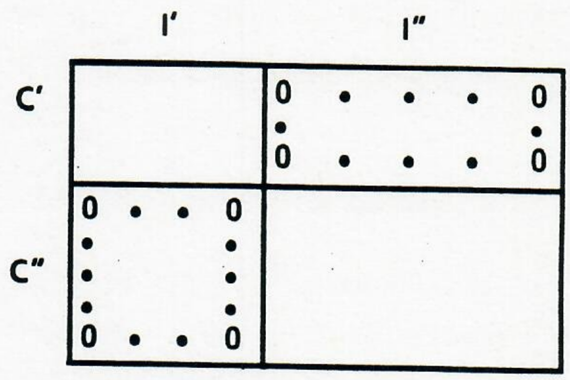

Figure 4

Further, either C' and I" are both non-empty, or C" and I' are both non-empty (i.e., at least one block of zeros in Figure 4 is really there).

Suppose the former. Let P, W > 0 be a price system sustaining S. Consider $\lambda = \text{Min}(W_i a_{ij}/P_j)$, the minimum taken over all $i \in C'$ and $j \in I"$. By (4), $\lambda \geq 1$. Change the price system by multiplying all P_j in I" and W_i in C" by λ (leaving P_j in I' and W_i in C' unchanged). This preserves (4) and (5), but transforms (4) to an equality at any $i \in C'$ and $j \in I"$ at which the minimum is attained. Changing such an s_{ij} to 1 preserves viability, hence the original S was not maximal. A similar argument works if C" and I' are both non-empty.

(ii) "Maximal" <u>implies</u> "strictly sustainable":

Suppose viable S is not strictly sustainable. Then it has a price system P, W but with $s_{ij} = 0$ and $P_j = W_i a_{ij}$ for some i,j.

Let S' be the same as S, except that $s'_{ij} = 1$ for this particular pair. The same price system supports S', so S' is viable. Hence S is not maximal.

(iii) "Connected graph" and "strict sustainability" imply "maximal":

First, if S has a connected graph, there is at most one price system supporting it (up to a numéraire - i.e., a multiplicative constant). Let P', W'; P", W" be two supporting price systems, and let $P'_1/P''_1 = c$. Then, by (5), $W'_i/W''_i = c$ for all row nodes i connected to column node 1, then $P'_j/P''_j = c$ for all columns j joined to these, etc. By connectivity every node is eventually reached, and ratios everywhere equal c (cf. McKenzie, p. 169).

Now let viable S have a connected graph but not be maximal. Let viable S' be the same as S, except that $s_{ij} = 0$, $s'_{ij} = 1$ for some particular (i,j) pair. Let P, W be a price system supporting S', so that $P_j = W_i a_{ij}$ for this pair. The same P, W supports S, though (12) is violated at (i,j). But any price system for S is proportional to this one, hence S is not strictly sustainable. ▒

As corollaries, every maximal pattern S has at least m+n-1 "1"s, and also is efficient (no all-zero rows or columns). These follow from connectivity. For, a connected graph with m + n nodes has at least m + n - 1 arcs (Berge, p. 16). Further, if row i is all zeros, then country i is not connected to any other node. Similarly for columns.

I.6. Technologies in General Position

Technology A is <u>generic</u>, or <u>in general position</u>, if for every square submatrix the Jones match is unique (i.e., there are no ties for lowest score). Technologies that are not generic are said to be <u>exceptional</u>.

For example, in the classic 2-by-2 case (Figure 1) the technology is generic if ad $\neq$ bc, exceptional if ad = bc. The reason for making this distinction is that many results are very much simplified if exceptional technologies are excluded, while, at the same time, those technologies really are exceptional "knife-edge" cases.

A technology is identified by mn numbers, so we may think of the set of all possible technologies as (the positive orthant of) mn-dimensional space. Equip this space with Lebesgue measure. The following theorem states that the exceptional technologies are "very small" in several senses.

Theorem 5. (i) The set of exceptional technologies has measure zero.

(ii) The set of generic technologies is dense and open.

Proof. (i) An exceptional technology satisfies a relation of the form

$$a_1 a_2 \ldots a_r = a_{r+1} \ldots a_{2r} \quad (14)$$

where $a_1, \ldots, a_{2r}$ are certain distinct coefficients in it. Choosing one of the a's and solving for it in terms of all the others, we see that the set of technologies satisfying (14) is the graph of a continuous function in mn-space. By Fubini's theorem, the measure of this set can be expressed as the integral of the measure of its "sections" (Faden, p.86). These measures are all zero since the sections are singletons. Thus the measure of the set satisfying (14) is zero. Finally, there are just a finite number of possible relations of the form (14), and the union of the corresponding sets has measure zero and contains the exceptional technologies.

(ii) Any non-empty open set has positive measure, hence contains a generic technology, by Part (i): Generics are dense. A generic technology satisfies a system of <u>in</u>equalities of the form (14) (replace "=" by "<"); a sufficiently small jiggling of the a_{ij}'s preserves these inequalities: Generics are open. ▒

Genericity can be characterized in a large number of ways by the properties of its viable patterns. The following theorems give the fundamental results.

Theorem 6. For technology A, each of the following properties implies the other two:

(i) A is in general position.

(ii) Every viable pattern S has an acyclic graph.

(iii) Every viable pattern S is strictly sustainable.

Proof. (i) <u>implies</u> (ii): Let $(C_1, I_1, C_2, I_2, \ldots, C_r, I_r)$ be a cycle in the graph of pattern S. ($C_1, \ldots, C_r$ are distinct countries, $I_1, \ldots, I_r$ distinct industries). Then $\{(C_1, I_1), \ldots,$

$(C_r, I_r)\}$ is a match in S, and so is $\{(C_1, I_r), (C_2, I_1), \ldots, (C_r, I_{r-1})\}$. By A generic these cannot both be Jones matches. Thus the McKenzie-Jones principle is violated, so S is not viable.

(ii) implies (iii):

Step 1: We first prove the following: if S is strictly sustainable, then the pattern S' that results from changing a single $s_{ij} = 1$ to 0 remains strictly sustainable.

S is acyclic, so the removal of arc (i,j) makes the graph of S' disconnected. Thus there is a partition of countries into sets C', C", and of industries into I', I", with no arc from C' to I" or from C" to I' (see Figure 4). We may also assume that i $\in$ C', j $\in$ I" (e.g., by taking C', I' to be the connected component containing i.)

Let P, W be a price system strictly sustaining S. Transform it as follows: multiply all W's and P's in C', I' by λ; leave all W's and P's in C", I" unchanged. Here $\lambda > 1$, but chosen so close to 1 that all strict inequalities of the form (12) are preserved. As for the equalities of the form (13) they are all preserved too, due to the absence of arcs noted above – with the sole exception of the special pair (i,j). Thus the new price system strictly sustains S', and Step 1 is finished.

Step 2: Viable S extends to a maximal pattern S", which is strictly sustainable (Theorem 4). Now start from S", and change "1"s back into "0"s one at a time until we get back to S. By

Step 1 strict sustainability is preserved at each step, so that S itself is strictly sustainable.

(iii) <u>implies</u> (i): Suppose A is exceptional. Then there are distinct Jones matches in the same square submatrix – say $\{(1,\sigma(1)),\ldots,(r,\sigma(r))\}$ and $\{(1,\tau(1)),\ldots,(r,\tau(r))\}$ where σ, τ are distinct bijections onto the same set of industries. Define pattern S as follows: $s_{i,\sigma(i)} = 1$, $i = 1,\ldots, r$, and zeros elsewhere. S is viable. Let P, W > 0 be a price system for S. Then

$$P_{\tau(i)} \le W_i a_{i\tau(i)}\ ,\ i = 1,\ldots, r \qquad (15)$$

by (4). We then obtain

$$P_{\sigma(1)}\ldots P_{\sigma(r)} = P_{\tau(1)}\ldots P_{\tau(r)} \le W_1\ldots W_r a_{1\tau(1)}\ldots a_{r,\tau(r)} \qquad (16)$$
$$= W_1\ldots W_r a_{1\sigma(1)}\ldots a_{r,\sigma(r)}$$

(The first equality in (16) arises from σ,τ mapping to the same set of industries, the inequality from multiplying all the inequalities in (15), and the last equality from the Jones matches having tied scores). But $P_{\sigma(i)} = W_i a_{i,\sigma(i)}$, $i = 1,\ldots, r$, by (5). Multiplying these shows that (16), hence (15), is actually all equalities. On the other hand, $s_{i,\tau(i)} = 0$ for some i, since σ, τ are distinct. Thus S is not strictly sustainable, and (iii) is false. ▒

Another set of necessary and sufficient conditions arise in connection with maximal patterns.

Theorem 7. For technology A of size (m,n), each of the following conditions implies all the others:

(i) A is generic.

(ii) The graph of every maximal pattern S is a tree.

(iii) Every maximal pattern S has exactly m + n - 1 "1"s.

(iv) Every viable pattern S with exactly m + n - 1 "1"s is maximal.

(v) Every viable pattern S whose graph is a tree, is maximal.

Proof. (ii) implies (iii), and (iv) implies (v): A tree on m + n nodes has exactly m + n - 1 arcs (Berge, p. 24).

(iii) implies (iv): Let (iv) be false, so that some S with m + n - 1 "1"s is viable, but not maximal. Change some "0"s to "1"s to obtain a maximal S'. This contradicts (iii).

(v) implies (ii): Let (ii) be false, so that some maximal pattern S has a graph with cycles (and also connected of course). A proper subset of the arcs forms a tree (Berge, p. 25). The corresponding pattern S is not maximal, falsifying (v).

(i) if and only if (ii): (ii) says that every maximal pattern is acyclic. Hence every viable pattern S is acyclic, since S arises from some maximal S' by changing some "1"s to "0"s. Apply Theorem 6. ▒

Finally, Theorems 8 and 9 below give the basic uniqueness results associated with generic technologies.

Theorem 8. For technology A of size (m, n), each of the following conditions implies the other two:

(i) A is generic.

(ii) Let $X_1, \ldots, X_n \geq 0$ be output efficient for $L_1, \ldots, L_m > 0$. Then there is <u>exactly one</u> system of allocations (L_{ij}), (X_{ij}) satisfying (1), (2), (3).

(iii) Let $L_1, \ldots, L_m \geq 0$ be input efficient for $X_1, \ldots, X_n > 0$. Then the same conclusion holds.

Proof. (i) <u>implies</u> (ii): Let (ii) be false. Then two distinct allocations (X'_{ij}), (X''_{ij}) satisfy (1), (2), (3). Let $\chi_{ij} = X'_{ij} - X''_{ij}$. Then

$$\Sigma_i\, \chi_{ij} = 0, \text{ all } j, \quad \Sigma_j\, a_{ij}\chi_{ij} = 0, \text{ all } i$$

In the χ matrix there is a "+" entry, then a "-" entry in the same column, then a "+" entry in the same row, etc. Eventually we get to a cycle of labeled entries as in Figure 3. $X'_{ij} > 0$ at all "+" entries, and $X''_{ij} > 0$ at all "-" entries.

Now consider $X_{ij} = (X'_{ij} + X''_{ij})/2$. This also satisfies (1), (2) and (3), and is positive at <u>all</u> the labeled entries in Figure 3. Hence the graph of its pattern has a cycle, so (i) is false.

(ii) <u>implies</u> (i): Let (i) be false, so that there exists a viable pattern S whose graph contains a cycle, say $(I_1, C_1, I_2, C_2, \ldots, I_r, C_r)$ (Figure 5). We may assume S has no all-zero rows.

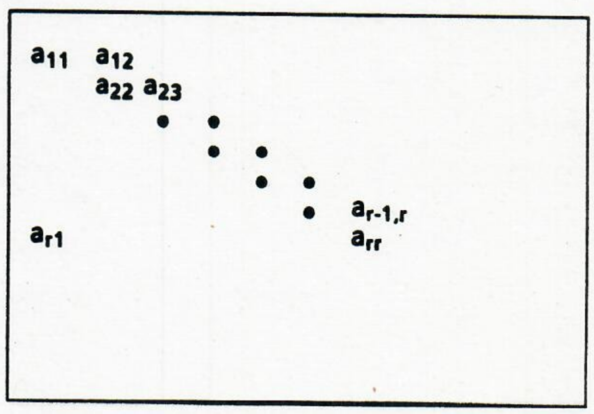

Figure 5

Let (L_{ij}) (and (X_{ij})) be allocations corresponding to S: $L_{ij} > 0$ if and only if $s_{ij} = 1$. Now alter (L_{ij}) by increasing L_{11} and decreasing L_{12} by a small amount ϵ, increasing L_{22} and decreasing L_{23} by $\epsilon a_{22}/a_{12}$ (these changes add and subtract equal amounts from X_2, leaving it unaltered), etc., around the cycle. (We are simplifying notation by writing C_i as i, I_j as j). An easy calculation shows that all $L_1, \ldots, L_m$, $X_1, \ldots, X_n$ are unaltered. This follows from the equality $a_{11} \ldots a_{rr} = a_{12} \ldots a_{r-1,r} a_{r1}$, these being the scores of tied Jones matches. Thus (ii) is false. (This argument is the same as McKenzie's involving "neutral circuits" (p. 170-71); also Jones, p. 169-70)).

(i) <u>implies</u> (iii), (iii) <u>implies</u> (i): Proofs are virtually identical to the above. ▒

Note that (ii) and (iii) are themselves virtually identical. In fact $X = (X_1, \ldots, X_n) > 0$ is output-efficient for $L = (L_1, \ldots, L_m) > 0$ if and only if L is input efficient for X, as is easily seen.

In the following, and throughout this paper, "natural number" includes zero.

Theorem 9. Let A be a technology of size (m,n). A is in general position if and only if, for any natural numbers $(r_1, \ldots, r_m)$ and $(c_1, \ldots, c_n)$, there is <u>at most one</u> viable pattern S having these as its row and column signatures.

Proof. <u>Only if</u>. Let S', S" be two distinct viable patterns with the same row and column signatures. The matrix S'- S" has all row and column sums zero. We can then find a cycle of alternating "+" and "-" entries as in Figure 3. At the "+" entries, $s'_{ij} = 1$, so these constitute a match in S'. Similarly, $s''_{ij} = 1$ at the "-" entries, giving a match in S". By McKenzie-Jones these are both Jones matches. But they are in the same square submatrix, so A is exceptional.

__If__. Let, say, $\{(1,\sigma(1)),\ldots, (r,\sigma(r))\}$, and $\{(1,\tau(1)),\ldots, (r,\tau(r))\}$ be distinct Jones matches, S_σ the pattern with "1"s at $(i,\sigma(i))$, $i = 1,\ldots, r$, "0"s elsewhere, and S_τ the corresponding pattern for τ. These are viable and distinct, with the same row signatures, hence distinct column signatures. Thus A is generic. ▒

I.7. Patterns and Facets

There are far-reaching structural connections between the algebraic and geometric representations of patterns. Let $L_1,\ldots, L_m > 0$ be given, as well as $P = (P_1,\ldots, P_n) > 0$. F_p, the __facet determined by__ P, is the set of all $(X_1,\ldots, X_n) \geq 0$ maximizing PX subject to (1), (2) and (3). By linear programming theory (Murty, p. 139) F_p is a convex, closed, polyhedral set.

P also determines uniquely a pattern S_p that is output efficient and strictly sustained by P and the corresponding $W_i = \mathrm{Max}_j[P_j/a_{ij}]$, $i = 1,\ldots,m$, namely $s_{ij} = 1$ if $P_j = W_i a_{ij}$, and 0 otherwise.

An allocation $L_{ij} \geq 0$ satisfying (1) __conforms__ to pattern S if $L_{ij} > 0$ implies $s_{ij} = 1$, all i,j. Any allocation generates an output n-tuple $(X_1, \ldots, X_n)$ via (2) and (3). The following theorem holds for any technology A.

Theorem 10. Let X $(=X_j)$ be generated by allocation (L_{ij}). (L_{ij}) conforms to S_p if and only if X belongs to F_p.

Proof. Let $X \in F_p$. Then allocation (L_{ij}) solves a linear program whose solution has already been described: If $L_{ij} > 0$, then P_j/a_{ij} attains the maximum over j, namely W_i. Hence $s_{ij} = 1$ by definition of S_p. Thus (L_{ij}) conforms to S_p.

Conversely, let allocation (L_{ij}) conform to S_p, and let X' be generated by any other allocation (L'_{ij}). Then

$$PX = \Sigma_j\Sigma_i\ L_{ij}P_j/a_{ij} = \Sigma_j\Sigma_i\ L_{ij}W_i = \Sigma_i\ L_iW_i =$$
$$\Sigma_j\Sigma_i\ W_iL'_{ij} \geq \Sigma_j\Sigma_i\ P_jL'_{ij}/a_{ij} = PX'$$

(The inequality arises from (4), the second equality from $L_{ij} > 0$ implying $s_{ij} = 1$ and (5) or (13); the other equalities from (1), (2) and (3). This shows that $X \in F_p$. ▒

As a corollary, F_p is the set of all n-tuples generated by allocations conforming to S_p. Furthermore, while X may be generated by more than one allocation, for any S_p they either all conform or all disconform.

For patterns S, S', the relation $S \leq S'$ means $s_{ij} = 1$ implies $s'_{ij} = 1$.

Theorem 11. For any price systems P and P', $S_p \leq S_{p'}$ if and only if $F_p \subseteq F_{p'}$.

Proof. If. Let, say, $(S_p)_{11} = 1$, and let (L_{ij}) be an allocation conforming to S_p with $L_{11} > 0$. If X is generated by this allocation, then $X \epsilon F_p$, hence $X \epsilon F_{p'}$, hence (L_{ij}) conforms to $S_{p'}$, hence $(S_{p'})_{11} = 1$.

Only if. Let $X \epsilon F_p$ and let X be generated by (L_{ij}). Then this allocation conforms to S_p, hence to $S_{p'}$, hence $X \epsilon F_{p'}$. ▒

Now, pattern S is of the form S_p if and only if S is strictly sustainable and output-efficient. The above discussion demonstrates a natural 1-to-1 correspondence between the set of these patterns and the set of facets, namely S_p corresponds to F_p. For, from Theorem 11, $S_p = S_{p'}$ if and only if $F_p = F_{p'}$, so the correspondence does not depend on the particular p representing a pattern or facet.

For generic technology A, every viable pattern is strictly sustainable, so the 1-1 correspondence is between the set of all facets and the set of all output-efficient patterns.

I.8. Dimensionality

We can also determine the dimensionality of facets from their patterns. Let output efficient S have r_i "1"s in row i. This gives $r_i - 1$ degrees of freedom to the range of conforming allocations. Adding over rows yields k - m degrees of freedom in all, where k is the number of "1"s in S. This is the dimensionality of the polyhedron of conforming allocations.

The mapping from allocations to output n-tuples is a linear transformation. If A is in general position this mapping is injective, so that it preserves dimensionality. Thus the dimension of the S-facet is also k - m. (For <u>any</u> technology, the dimension of F_p is given by McKenzie, p. 173, as n minus the number of connected components in the graph of S_p. If S_p has an acyclic graph, this equals k - m (McKenzie, p. 175) in agreement with our argument).

For example, consider maximal patterns. These have exactly m + n - 1 "1"s, hence their facets have dimension n-1, evidently the maximum possible in n-space. Thus the maximal patterns correspond exactly to the maximal facets, which we shall call <u>faces</u>. At the other extreme consider a viable S having exactly one "1" in each row (each country specializes in just one industry). Clearly there is just one conforming allocation, yielding a 0-dimensional facet of one point. We shall call this, and the corresponding pattern, a <u>vertex</u>.

The analysis above is in terms of output efficiency. There is a completely parallel analysis for <u>input</u> efficiency, which we sketch briefly. Given $X_1, \ldots, X_n > 0$, and $W_1, \ldots, W_m > 0$, the <u>input facet</u> F_W <u>determined by</u> W is the set of all L = $(L_1, \ldots, L_m)$ minimizing WL subject to (1), (2) and (3). S_W is <u>input</u>-efficient (no all-zero columns), given by s_{ij} = 1 if and only if $P_j = W_i a_{ij}$, where $P_j = \text{Min}_i[W_i a_{ij}]$. Then F_W is the set of all m-tuples generated by allocations conforming to S_W. Next, the set of strictly sustainable input-efficient patterns is in natural 1-1 correspondence with the set of input facets. For generic A, the dimension of an S-input facet is k - n, where k is the number of "1"s in S. Maximal patterns again correspond to maximal input facets, all of dimension m-1 (<u>input faces</u>). Finally, an <u>input vertex</u> is a viable pattern with exactly one "1" in each column (each industry operates in just one country).

I.9. Vertices

A facet is determined by its vertices (it is their convex hull). In the same way patterns are built up from the vertices contained in them. The following arguments work with any technology A (m-by-n, as usual).

Let S be a pattern with no all-zero rows. A <u>vertex in</u> S is a vertex S' ≤ S. That is, in each row we keep exactly one of the "1"s, setting the others to zero so the row signature is all "1"s. If r_i is the number of "1"s in row i, then the number of vertices in S is the product $r_1 r_2 \ldots r_m$.

Theorem 12. S is viable if and only if every vertex in S is viable.

Proof. <u>Only if</u>. Removing "1"s preserves viability.

<u>If</u>. Suppose S is not viable. Then there is a match in S that is not a Jones match, by generalized comparative advantage. By appropriate labeling we may take this match to be {(1,1),..., (r,r)}. For each row i other than 1,..., r choose any j with s_{ij} = 1. These m - r pairs, together with the r match pairs, yield a vertex S' in S. {(1,1),...,(r,r)} is also a match in S', so S' is not viable. ▒

(This result is stated in McKenzie (1968), p. 96, based presumably on McKenzie (1954), p. 171).

A caution. Let S', S" be viable vertices. Their "combination" S = Max(S', S") is not necessarily viable, since there may be more vertices in S than the original two. For example, England and Portugal may both specialize in wine, or they may both specialize in cloth. These are both viable vertices, but their combination is not viable in general.

Theorem 13. Given technology A of size (m, n), and given natural numbers $(c_1, \ldots, c_n)$ adding up to m:

(i) There exists a viable vertex with this column signature.

(ii) If A is generic, there is just one such vertex.

Proof. (i) Consider the transportation problem (10) with $r_i = 1$, $i = 1, \ldots, m$. Since $\Sigma_i r_i = \Sigma_j c_j$, this program has an optimal solution which is, furthermore, all integer (Murty, p. 382). Letting (s_{ij}) be this solution, we see that $s_{ij} = 0$ or 1 (S is a pattern) and has row signature (1,..., 1) (a vertex) and column signature $(c_1, \ldots, c_n)$. By Theorem 2, S is viable.

(ii) Immediate from Theorem 9. ▒

Theorem 14. S is a viable vertex if and only if it has minimal score over all other vertices with the same column signature.

Proof. The "only if" part is immediate from Theorem 3. Conversely, let S not be viable. Then it does not optimize the transportation problem (10) determined by its row and column signatures. But some other vertex S' does optimize this problem, so that $\Sigma_i\Sigma_j\ t_{ij}s_{ij} > \Sigma_i\Sigma_j\ t_{ij}s'_{ij}$. Taking antilogs shows that S does not attain minimal score. ▒

This is stated by Jones, p. 168. For the special case on which he concentrates attention, take m = n, with $(c_1,\ldots, c_n)$ all "1"s (cf. the discussion following Theorem 3).

Needless to say, this entire development with vertices has a parallel with input vertices, in which the roles of rows and columns are interchanged. Discussion is omitted.

I.10. Pattern Accounting

Theorem 13 on vertices is part of a very general class of results concerning the number of distinct patterns or facets in terms of their signatures or dimensions. It is remarkable that these results do not depend on the technology, provided only that it is generic. (How these facets fit together does depend on the technology, and provides a basis for classifying technologies).

We begin with the opposite extreme from vertices, the maximal patterns. Recall that, for A generic of size (m,n), these have exactly m + n - 1 "1"s.

Theorem 15. Let technology A be in general position.

(i) Let $r_1,\ldots, r_m$ be positive integers adding up to m + n - 1; then there exists exactly one viable pattern with this row signature.

(ii) Let $c_1, \ldots, c_n$ be positive integers adding to $m + n - 1$; there exists <u>exactly</u> <u>one</u> viable pattern with this column signature.

Proof. The hardest part is to prove that there is <u>at most one</u> such pattern. Steps 1 through 5 address this task.

Step 1: Let S', S" be two maximal patterns, sustained by prices P', W', P", W", respectively. We show

$$\mathrm{Min}_i(W''_i/W'_i) = \mathrm{Min}_j(P''_j/P'_j) \qquad (17)$$

Let i be a row attaining the left-hand minimum in (17), and let j be a column such that $s'_{ij} = 1$ for this i. (There is such a j, by S' maximal). Then $W''_i a_{ij} \geq P''_j$, $W'_j a_{ij} = P'_j$ by (4), (5), respectively, implying $W''_i/W'_i \geq P''_j/P'_j$ This proves (17) with "≥" in place of "=".

Conversely, let j be a column attaining the right-hand minimum in (17), and let i be a row such that $s''_{ij} = 1$ for this j. Then $W''_i a_{ij} = P''_j$, $W'_i a_{ij} \geq P'_j$ by (5), (4), respectively, implying $W''_i/W'_i \leq P''_j/P'_j$, implying (17) with "≤" in place of "=". Thus (17) is proved.

Step 2: Let C_0 be the set of countries i attaining the minimum in (17), and I_0 the set of industries j attaining the minimum. Let C_1 and I_1 be the remaining countries and industries, respectively. Suppose C_1 were empty; then the wage systems W', W" would be proportional to each other, implying S' = S". Suppose I_1 were empty; then P', P" would be proportional to each other, again implying S' = S".

The remaining possibility is that C_0, C_1, I_0, I_1 are all non-empty. We shall show in this case that S' or S" has a disconnected graph, contradicting its maximality. Specifically we show there is no arc joining i in C_0 to j in I_1 and no arc joining C_1 to I_0 (cf. Figure 4).

Step 3: Let i be in C_0, j in I_1 and suppose $s'_{ij} = 1$. Then $(W''_i/W'_i) < (P''_j/P'_j)$ by (17), $W'_i a_{ij} = P'_j$ by (5), and $P''_j \le W''_i a_{ij}$ by (4). Multiplication yields a contradiction, so that $s'_{ij} = 0$ for i in C_0, j in I_1.

A similar argument shows that $s''_{ij} = 0$ for i in C_1, j in I_0.

Step 4: Suppose i is in C_0, j in I_0. Then $P'_j/W'_i = P''_j/W''_i$ by (17). If this common value $= a_{ij}$, then $s'_{ij} = 1 = s''_{ij}$, by maximality of S', S". If this value $< a_{ij}$, then $s'_{ij} = 0 = s''_{ij}$. In any case $s'_{ij} = s''_{ij}$.

Step 5: Suppose now that S', S" have the same <u>row</u> signature. If i is in C_0, this implies $s''_{ij} = 0$, all j in I_1, from the results of Steps 3 and 4. But then S" has a disconnected graph, contradiction.

Suppose instead that S', S" have the same <u>column</u> signature. If j is in I_0, this implies $s'_{ij} = 0$, all i in C_1. This, together with step 3, shows that S' has a disconnected graph, contradiction. Thus S' = S". We have shown there is <u>at most one</u> maximal pattern with given row or column signature.

Step 6: For technology of size m-by-n, let $M_{m,n}$ be the set of maximal patterns, let $K_{m,n}$ be the set of all n-tuples $(c_1, \ldots, c_n)$

of positive integers adding up to m + n - 1, and let $L_{m,n}$ be the set of all m-tuples $(r_1, \ldots, r_m)$ with the same property. Let"card" be the cardinality of a set. Then

$$\text{card } K_{m,n} = \text{card } L_{m,n} \tag{18}$$

For, card $K_{m,n}(L_{m,n})$ equals the number of ways of putting m + n -1 indistinguishable balls into n(m) urns, with no urn remaining empty. Both these numbers equal (m + n - 2)! / [(m-1)! (n-1)!] (Feller, p. 37).

Step 7: The map assigning each maximal pattern its row (column) signature sends $M_{m,n}$ into $L_{m,n}$ ($K_{m,n}$), respectively. These two maps are injective by Step 5, since distinct patterns have distinct signatures.

Step 8: Finally, we prove the theorem by induction on t = m + n. To start, t = 2 is the 1-by-1 case, for which the theorem is trivial. Suppose, then, that the theorem holds for all (m, n) such that m + n = t - 1, and consider (m, n) with m + n = t.

Case (i): $m \le n$. Let $c_1, \ldots, c_n$ be positive integers adding to m + n - 1. If all $c_j \ge 2$, we would have $m + n - 1 \ge 2n$, contradiction. Hence some $c_j = 1$, say $c_1 = 1$. Drop this column from the technology. The result is a technology of size (m, n - 1), still generic. $(c_2, \ldots, c_n)$ adds up to m + (n - 1) - 1, hence, by the induction hypothesis, there exists a (unique) maximal pattern S of size (m, n - 1) with column signature $(c_2, \ldots, c_n)$. Let $W_1, \ldots, W_m$ be a wage system for it. Now go back to the (m, n) technology and define pattern S' on it as follows: S' coincides with S in columns 2 through n, while in column 1 $s'_{i1} = 1$ on the i minimizing $W_i a_{i1}$, and = 0 elsewhere (there cannot be a tie, by genericity). It is easily verified that this S' is a maximal pattern with column signature $c_1, \ldots, c_n$.

Case (ii): $m \geq n$. An identical argument, reversing the role of rows and columns, shows that for any positive integers $r_1, \ldots, r_m$ adding to $m + n - 1$ there exists a maximal pattern with this row signature.

Step 9: Case (i): $m \leq n$. Steps 7 and 8 together show that the mapping from $M_{m,n}$ to $K_{m,n}$ is a bijection or 1-to-1 correspondence. Hence card $M_{m,n}$ equals the common value in (18). But then the injection from $M_{m,n}$ to $L_{m,n}$ must also be a bijection, so that for every $(r_1, \ldots, r_m)$ as above there is a (unique) maximal pattern with that row signature. The induction is now complete in this case.

Case (ii): $m \geq n$. Identical argument, reversing the roles of $K_{m,n}$ and $L_{m,n}$. The induction is now complete. ▒

Incidentally, steps 1 through 5 in this proof do not assume A generic, so we have also proved: There is at most one maximal pattern with a given row <u>or</u> column signature, even if A is exceptional.

We now come to what may be called the <u>General Facet Accounting Theorem</u>. For convenience, we work with the output-efficiency frontier, but a similar result holds for input efficiency. We use the standard abbreviation $\binom{r}{s}$ for $r!/[s!(r-s)!]$ for natural numbers r, s.

Theorem 16. Let technology A of size (m, n) be generic, and let $r_1, \ldots, r_m$ be positive integers adding to k.

(i) The number of viable patterns with row signature $(r_1, \ldots, r_m)$ is

$$\binom{m + n - 1}{k} \tag{19}$$

(ii) The number of efficient patterns with row signature $(r_1, \ldots, r_m)$ is

$$\binom{m - 1}{k - n} \tag{20}$$

Proof. The proof is by induction on $d = k - m$. (Recall that d is the dimensionality of the facet associated with a pattern with k "1"s and m rows).

Step 1: Start with $d = 0$. Here $(r_i, \ldots, r_m) = (1, \ldots, 1)$. For any natural numbers $c_1, \ldots, c_n$ adding up to m there exists exactly one viable vertex with column signature $(c_1, \ldots, c_n)$ (Theorem 13). Hence the number of viable patterns with row signature $(1, \ldots, 1)$ equals the number of $(c_1, \ldots, c_n)$ adding to m, which is (Feller, p. 36)

$$\binom{m + n - 1}{m}$$

This verifies (19) for $d = 0$. The number of <u>efficient</u> vertices (no all-zero columns) equals the number of <u>positive</u> integers $(c_1, \ldots, c_n)$ adding to m, which is

$$\binom{m-1}{m-n}$$

verifying (20) for d = 0. Thus the theorem holds for d = 0.

Step 2: Suppose, then, that the theorem holds for dimension d (i.e., for any generic technology A of any size (m, n) and any k with k - m = d), and that we are given generic A and positive integers $r_1, \ldots, r_m$ adding to k, where k - m = d + 1.

At least one of the r_i exceeds one, say $r_1 > 1$. We now construct a new technology A' of size (m + 1, n) as follows. Rows 1 through m of A' are the same as A. The new row — call it row 0 — is almost, but not quite, proportional to row 1. Specifically, we require that A' remain in general position, and that

$$a_{0j}a_{1\ell}/(a_{1j}a_{0\ell}) < \lambda \tag{21}$$

for all j, ℓ = 1,..., n. Here λ is a number > 1 but very close to 1, satisfying the following conditions:

(i) For each square submatrix of size (r, r) in A (r > 1), let ρ be the ratio of the score of the second-best match to the score of the Jones match ($\rho > 1$ by A generic). λ is to be less than all such ρ.

(ii) For each viable pattern S in A with row signature $(r_1-1, r_2, \ldots, r_m)$ <u>or</u> $(r_1, r_2, \ldots, r_m)$ choose a price system $W_1, \ldots, W_m, P_1, \ldots, P_n$ <u>strictly</u> sustaining S; then consider $\rho = \text{Min}[W_1a_{1j}/P_j]$, the minimum taken over all j = 1,..., n for which $s_{1j} = 0$ ($\rho > 1$ by (12)). λ is to be less than all such ρ.

These are the conditions on row 0. To fulfill them, start by duplicating row 1, then "jiggle" each coefficient successively to attain general position (cf. Theorem 5; details are omitted).

Step 3: We need one preliminary result. Let S be a viable pattern with row signature either $(r_1, \ldots, r_m)$ or $(r_1-1, r_2, \ldots, r_m)$, and let P, W be the price system for S that was used in determining λ above. Let j, ℓ be two columns such that $s_{1j} = 0$, $s_{1\ell} = 1$. Then $P_j/a_{0j} < P_\ell/a_{0\ell}$.

This follows from

$$a_{0\ell}a_{1j}/(a_{0j}a_{1\ell}) < \lambda < W_1 a_{1j}/P_j = (P_\ell/a_{1\ell})(a_{1j}/P_j),$$

from (21) and the construction of λ.

Step 4: For technology A' consider the m + 1 tuple $(1, r_1-1, r_2, \ldots, r_m)$ of positive integers. This adds up to k, and $k - (m + 1) = d$. Hence, by the induction hypothesis, there exist exactly $\begin{bmatrix} m+n \\ k \end{bmatrix}$ viable patterns S' in A' with this row signature. Each of these has exactly one "1" in row 0, say in column ℓ.

Now classify these patterns into two groups:

Group I patterns: satisfy $s'_{1\ell} = 1$

Group II patterns: satisfy $s'_{1\ell} = 0$

Step 5: We show there is a 1-to-1 correspondence between Group I and the set of all viable patterns S with row signature $(r_1-1, r_2, \ldots, r_m)$. Given such an S, define S' as follows. S' has an extra row 0, with a single "1" in the column ℓ maximizing P_j/a_{0j} over $j = 1, \ldots, n$, where the P's are from the price system for S used above. In rows $1, \ldots, m$, S' is identical to S.

By Step 3, $s_{1\ell} = 1$. (Also, there cannot be a tie for maximum, by A' generic). S' is viable, since supported by the price system P, W (with $W_0 = P_\ell/a_{0\ell}$). Thus S' is in Group I. This is indeed a 1-1 correspondence, for, given S' there is only one S it can have arisen from (delete row 0), and this S is viable.

Step 6: We show there is a 1-to-1 correspondence between Group II and the set of all viable patterns S with row signature $(r_1,\ldots, r_m)$. Given such an S, define S' exactly as in Step 5, <u>except</u> that we set $s'_{1\ell} = 0$. ($s_{1\ell} = 1$ by the argument of Step 3.) Thus S' is in Group II.

Given S', there is only one S it can have arisen from (delete row 0 and change $s'_{1\ell} = 0$ to a "1"). It remains only to show that this S is viable. If S were not viable, then some match in it must not be a Jones match, and this offending match must arise from the extra "1" at $(1, \ell)$ (since S' is viable). Label countries and industries so that $(2,2),\ldots, (r,r)$ are the remaining components of this bad match, while the corresponding Jones match is $\{(1,2),\ldots, (r-1,r), (r,\ell)\}$. Thus (see Figure 5 with column ℓ in place of column 1)

$$a_{1\ell}a_{22}\cdots a_{rr} > a_{12}\cdots a_{r-1,r}a_{r\ell} \tag{22}$$

We can, in fact, multiply the right side of (22) by λ and still retain the inequality, by construction of λ. On the other hand, $\{(0,\ell), (2,2),\ldots, (r,r)\}$ is a match in viable S', hence a Jones match. Hence

$$a_{02}a_{23}\cdots a_{r-1,r}a_{r\ell} > a_{0\ell}a_{22}\cdots a_{rr} \tag{23}$$

Multiply the inequalities (23) and (22) (with λ included on the right side of (22)). This yields after cancellations, $a_{02}a_{1\ell} > a_{12}a_{0\ell}\lambda$, which contradicts (21). Thus S is viable after all, and the 1-to-1 correspondence is established.

Step 7: We now complete the induction for Part (i) of the theorem. The number of viable patterns in A with row signature $(r_1, \ldots, r_m)$ = the number of Group II patterns (Step 6)

$$= \begin{pmatrix} m+n \\ k \end{pmatrix} - \text{number of Group I patterns (Step 4)}$$

$$= \begin{pmatrix} m+n \\ k \end{pmatrix} - \text{number of viable patterns in A with}$$

signature $(r_1-1, r_2, \ldots, r_m)$ (Step 5)

$$= \begin{pmatrix} m+n \\ k \end{pmatrix} - \begin{pmatrix} m+n-1 \\ k-1 \end{pmatrix} = \begin{pmatrix} m+n-1 \\ k \end{pmatrix} \quad (24)$$

verifying (19). Only the next-to-last equality needs explaining. $(r_1-1, r_2, \ldots, r_m)$ adds up to $k - 1$, and $(k - 1) - m = d$, so the induction hypothesis can be applied.

Step 8: To complete the induction for Part (ii) of the theorem. Substitute "efficient" for viable throughout, and note that the mappings in Steps 5 and 6 preserve the "no all-zero column" condition. The entire argument goes through. The structure of Step 7 is preserved, and the appropriate formula reads

$$\begin{pmatrix} m \\ k-n \end{pmatrix} - \begin{pmatrix} m-1 \\ k-n-1 \end{pmatrix} = \begin{pmatrix} m-1 \\ k-n \end{pmatrix}$$

in place of (24). This verifies (20). ▒

A few comments. If $n > k$, there are no efficient patterns since not all columns can be covered by "1"s. (20) gives the correct answer in this case, namely "zero". If $k = m + n - 1$, (19) and (20) both give the answer "one", in agreement with Theorem 15. (The one maximal pattern must be efficient). This result, arrived at here at the end of a long, somewhat tortuous induction, is arrived at there by a much shorter, perhaps more elegant, and completely different line of argument. It is important to develop a variety of techniques to gain insight into this complex subject.

Here are some simple corollaries.

Theorem 17. Let generic technology A be of size (m, n).

(i) The total number of efficient patterns having k "1"s is

$$\frac{(k - 1)!}{(k - m)!\ (k - n)!\ (m + n - k - 1)!} \qquad (25)$$

(ii) The total number of facets of dimension d in the output efficiency frontier is

$$\frac{(m + n - 1)!\ /\ (d + m)}{d!\ (n - d - 1)!\ (m - 1)!} \qquad (26)$$

Proof. The number of distinct sequences of positive integers $(r_1, \ldots, r_m)$ adding up to k is $\binom{k - 1}{m - 1}$. This times (20) gives (25). This times (19) gives (26), on making the substitution $k = d + m$. ▒

Note some special cases. If $k = m + n - 1$, (25) gives the total number of maximal patterns as $(m + n - 2)!/[(m-1)!(n-1)!]$ which we already knew as the value of (18). The same answer arises from (26) when $d = n - 1$. If $m = n = k$, (25) equals one, the unique Jones match.

II. The Synthesis Problem for Three Industries or Three Countries

The astonishing regularities uncovered in pattern accounting are only half the story. The same collection of facets can be fitted together in many ways, and these alternatives <u>are</u> technology dependent – in fact they yield a natural approach to the classification of technologies.

This synthesis problem appears to be more difficult than the ones tackled up to this point. At any rate, we have made substantial progress only for the case of three industries, or three countries. There are good reasons for studying the $n = 3$ case. We can make full use of our geometrical intuition here, building on previous work by McKenzie. And even at this level there are rather deep unsolved problems, which give an inkling of what lies in store in higher dimensions.

II.1. Quincunxes and McKenzie Tilings

To McKenzie (1954), p. 174, goes the credit for constructing the first "McKenzie tiling" (in our terminology - see Figure 11 below). This was a remarkable insight. The derivation (p. 173) was by constructing edges connecting adjacent vertices – a method which is not always reliable and which in fact led to an historically-important error in their use (See Section I.3 above). Further, the general principles governing all such tilings were barely touched upon (p. 175).

Our aim here is to derive these general principles from the results of the analytic theory of Part I of this paper. We shall concentrate on the output efficiency frontier for the three industry case. The input efficiency frontier for the three country case is completely parallel, and will be mentioned from time to time. Assume until further notice that technology is in general position.

The facets consist of vertices, edges and faces (dimensions 0, 1, 2, respectively). The vertices and faces are in 1-1 correspondence with their column signatures, so we can use the latter as natural <u>codings</u>. Thus, a vertex is named by a triple (c_1, c_2, c_3) of natural numbers adding up to m (where m is the number of countries), c_j being the number of countries specializing in industry j. A face, on the other hand, has a signature (c_1, c_2, c_3) of <u>positive</u> integers adding up to m + n -1 = m + 2.

Consider the vertex signatures first. There is a natural way of arranging them, illustrated in Figure 6 for m = 4 countries. To facilitate discussion, refer to movements between adjacent vertices as being in the 1-2, 2-3 or 1-3 directions (Figure 7).

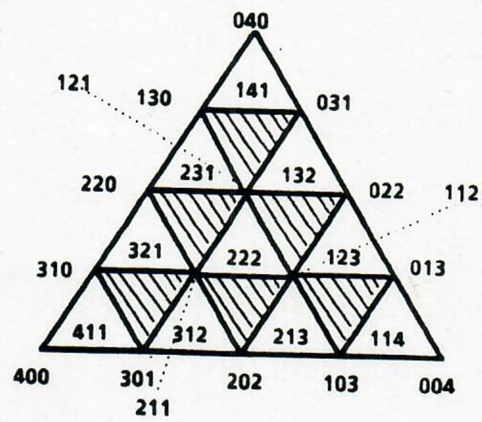

Figure 6

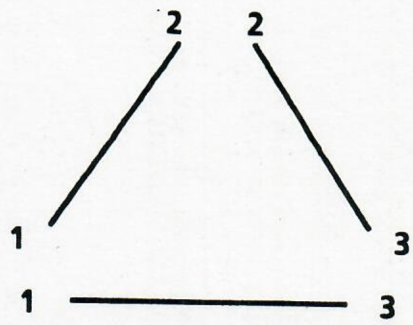

Figure 7

(Vertices (c_1, c_2, c_3) and (c_1', c_2', c_3') are <u>adjacent</u> if $\text{Max}|c_j - c_j'| = 1$). Movement in the 1-3 direction shifts one unit between column 1 and column 3, etc.

Three mutually adjacent vertices may be grouped into "triangles" in two distinct ways: <u>point-down triangles</u>, grouping (a-1, b, c), (a, b-1, c) and (a, b, c-1) (shaded in Figure 6), and <u>point-up triangles</u>, grouping (a-1, b-1, c), (a-1, b, c-1), and (a, b-1, c-1) (unshaded in Figure 6). Here (a, b, c) are positive integers adding up to m + 1 for point-down triangles, and to m + 2 for point-up triangles. The respective (a, b, c) triples may be used as code labels for these triangles (point-ups are labeled in Figure 6).

Call the entire configuration a <u>quincunx</u> (a 4-quincunx in Figure 6). It is purely schematic: There is no assumption that "adjacent" vertices are physically close, or connected by an edge, or that the triangles are actual faces.

Now consider the row signatures of faces. ($r_1, \ldots, r_m$) are positive integers adding up to m + 2. There are two types:

those of the form (1,1,..., 3,..., 1,1) with a single "3" in some row, and those of the form (1,1,..., 2,..., 2,...1,1) with two "2"s somewhere. The first type has 3 vertices, the second has 2·2 = 4 vertices.

Consider the particular face with signature (1,1,..., 3,..., 1,1), the "3" being in row i. Let (a,b,c) be its column signature. Row i has three "1"s, and changing any two of these to "0" yields a vertex. It follows that the three vertices have column signatures (a-1, b-1, c), (a-1, b, c-1), and (a, b-1, c-1). Hence <u>every face of this type is a point-up triangle</u>.

Consider the face with signature (1,1,..., 2,..., 2,..., 1,1), the "2"s being in rows h, i. First of all, the four "1"s in these rows must spread over all three columns, otherwise comparative advantage is violated. Figure 8 is an example. The vertices are specified by retaining one "1" in each row. Thus they may be labeled $\alpha\gamma$, $\beta\gamma$, $\beta\delta$ and $\alpha\delta$ in Figure 8. Comparison of their column signatures shows they must be arrayed as in Figure 9 on the quincunx.

h	1^α	1^β	0
i	0	1^γ	1^δ

Figure 8

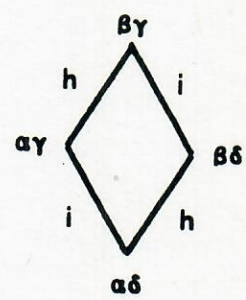

Figure 9

Now consider edges. These all have row signatures of the form (1,1,..., 2,..., 1,1) with a single "2" in some row. Two vertices are connected by an edge if and only if they are identical in all rows but one. This criterion yields the array of edges in Figure 9. Thus this face is a quadrilateral. Labeling each edge by the row in which it has two "1"s, we find that opposite sides get the same label (Figure 9). (The h-edges are in 1-2 direction, since $s_{h1} = 1 = s_{h2}$, etc.) For the triangular face considered above, all three edges are labeled with row i.

What about patterns other than Figure 8? Reversing rows h and i merely interchanges the edge labels in Figure 9. If the double "1"s occur in column 1 we get a quadrilateral arrayed as L in Figure 10. If they occur in column 3 we get an R-type quadrilateral, while column 2 yields a U-type as in Figure 9. All three types have identical structure, and can be obtained from each other by rotations of 120°. Triangular faces will be called type T. These four types exhaust the possible faces, under generic technology.

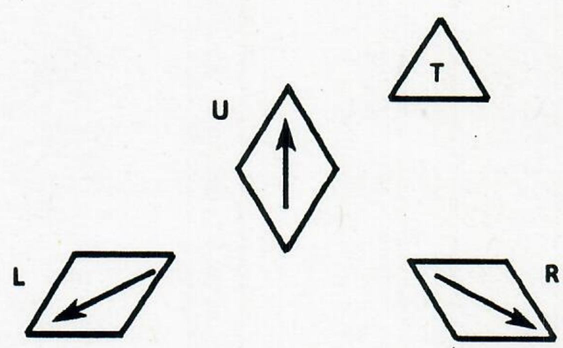

Figure 10

Note that each of the types, L, U, R consists of an "amalgamation" of a point-up and neighboring point-down triangle. Consider the set of all quadrilateral faces and the set of all point-down triangles; let σ be the map sending each face to its point-down component. We claim σ is injective. For, suppose first that two U-faces shared the same point-down component. Then the column signatures of all vertices of one would be the same as the other, hence the vertices themselves would be identical (Theorem 13), hence the faces would be identical; similarly for two L- or two R-faces. If two different types shared the same point-down component, then some pair of vertices both would and would not be connected by an edge, contradiction. Thus σ is injective.

Now consider the set of <u>all</u> faces and the set of all point-up triangles, and let τ be the map sending each face to its point-up component. A similar argument shows that τ is injective.

We now do some counting. The number of quadrilateral faces = the number of ways of selecting two objects out of m, which is $(m^2 - m)/2$. The number of point-down triangles = the number of ways of getting positive integers (a, b, c) to add up to m + 1, which is also $(m^2 - m)/2$. Hence σ is actually a bijection. Also, the total number of faces = $(m^2 + m)/2$ (from, say, (25) or by noting there are m T-faces). The number of point-up triangles = the number of ways of getting positive integers (a, b, c) to add to m + 2, which is also $(m^2 + m)/2$. Hence τ is also a bijection.

It follows that every configuration making up the output efficiency frontier of a generic technology may be described schematically as follows: Each point-down triangle is mapped to one of its three neighboring point-up triangles (with which it amalgamates to form a quadrilateral face; L-, R-, U-faces arise when the map is to the left, the right, or upward, respectively; see Figure 10). The mapping is to be injective, so this leaves exactly m point-up triangles unmated, and these become T-faces. Any configuration of this sort will be called a McKenzie tiling of the quincunx.

The easiest way to describe a McKenzie tiling is to specify the type of face (L, U, R, or T) assigned to each face column signature (c_1, c_2, c_3). An example is Figure 11, which is taken from McKenzie, p. 174 (with correction). This tiling in turn was constructed from one of Graham's (1948) world trade models involving three industries, Linen, Cloth and Corn, and four countries A, B, C, D. The vertex marked "Linen" is the one in which all countries produce linen, etc. Note that every edge is labeled by a country, as in Figure 9. The column signature of a face is always identical to the label of its point-up component as given in Figure 6. (This follows from the following general principle, whose proof is immediate. Let S of size (m, n) be any pattern with no all-zero row. The column signature of S is, componentwise, the maximum of

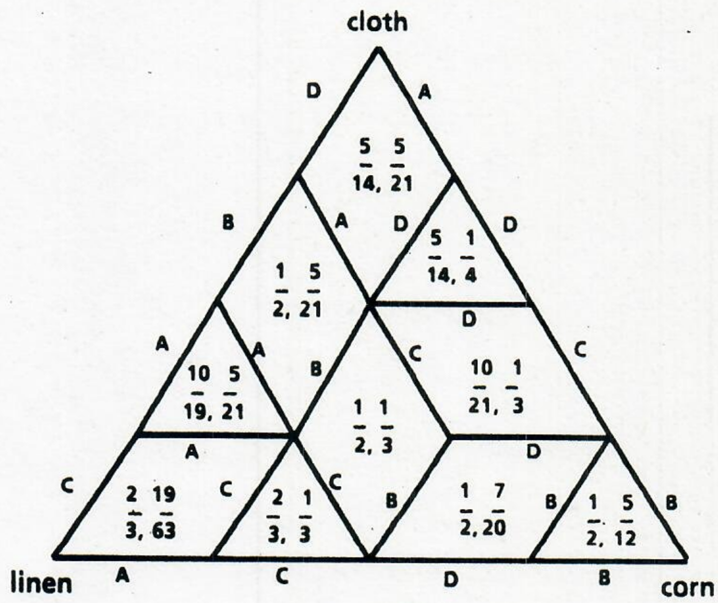

Figure 11

the column signatures of its vertices). Finally we have appended to each face a pair of numbers giving prices; this will be discussed later.

II.2. The Large Scale Structure of McKenzie Tilings

In a McKenzie tiling consider a T-face with its three edges having directions as in Figure 7. If the 1-2 edge is not on the quincunx border, the face on the other side of it must be either type L or type U. In this face, the edge opposite the original edge is again in direction 1-2. The same argument shows there must be a further L- or U-face on the other side of it. Thus we get a (uniquely determined) sequence of faces until the 1-2 border of the quincunx is reached, each face being either type L or U. Similarly, starting from the 2-3 edge of T, we get another sequence of faces to the 2-3 border, each being either type R or U. Finally, the 1-3 edge yields a third sequence of faces, each of type L or R. Figure 12 is an example.

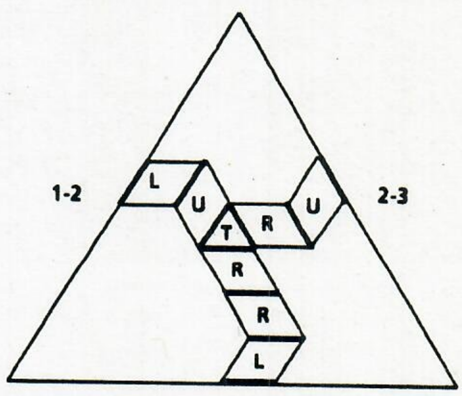

Figure 12

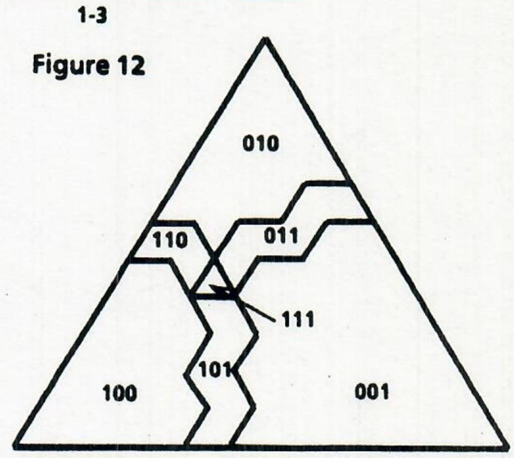

Figure 13

This construction determines a partition of the set of all faces into (at most) seven zones as in Figure

13. Zone 111 consists of T alone. Zones 110, 011, and 101 are the three sequences, respectively. Zones 100, 010 and 001 are the three remaining connected sets of faces touching vertices (m00), (0m0), (00m) respectively. (Some of these zones are missing if T has an edge on the quincunx border).

This zonation gives complete information about the i-th row of the pattern of every vertex, every edge, and every face in the tiling (where i is the row in which the pattern of T is 111), as follows: The i-th row of a vertex, edge or face in Zone xyz is xyz. (If a vertex or edge is on the borderline between several zones, the one with the smallest number of "1"s gives the correct row i).

To prove this, first consider the large-scale structure of edge labeling. The three edges of T get labeled by row i. By the argument used in constructing Figure 9, this labeling propagates along "parallel" edges in the three sequences of faces making up Zones 110, 011 and 101 (stressed in Figure 12). Furthermore, no other edge is labeled by row i — for, by "reverse parallel propagation" every other edge works back to a different triangular face. See Figure 11, for example. (If the tiling arises from a generic technology, this can also be demonstrated by a counting argument: The number of edges with row signature (1,1,..., 2,..., 1,1) with "2" in row i is m + 2 by (19). But if T has column signature (a, b, c), the number of edges in Zones 110, 101, 011 including the T-edge is c, b, a, respectively. Since a + b + c = m + 2, all such edges are accounted for).

Next, start say with vertex m00, for which row i is obviously 100. For any other vertex v take a sequence of vertices v_0, $v_1, \ldots,$ v_r where $v_r = v$, $v_0 = m00$, and each adjacent pair of vertices is connected by an edge in the tiling (there always is such a sequence). Now consider what happens to row i as we pass from vertex to vertex. If the connecting edge is not labeled by i, nothing happens. If it is labeled by i, the "1" jumps to a new column, as determined by the direction of the edge (Figure 7). It is then easily seen that the rule above is correct for vertices. Correctness for edges and faces then follows from the fact that row i for these elements is the maximum of the rows i for their component vertices.

This is a remarkable result. It says that a McKenzie tiling, labeled by rows, carries complete information about the patterns of all its facets (hence about all viable patterns, if the tiling arises from a generic technology).

II.3. The Physical Efficiency Frontier

A McKenzie tiling is of course purely schematic. The actual output efficiency frontier is a polygonal surface in 3-space, concave to the origin. (In the 3-by-n case, the input efficiency frontier also has this property but is convex to the origin). Can anything be said about the physical properties of this surface?

Theorem 18. (i) All quadrilateral faces are parallelograms.

(ii) Let e_1, e_2 be two edges of a face meeting at a vertex; if the angle they make in the tiling is (60°, 120°), then the actual physical angle is (acute, obtuse), respectively.

Proof. (i) It suffices to use Figures 8 and 9; all other cases are similar. Going from vertex $\beta\gamma$ to $\beta\delta$ involves shifting the entire labor pool L_i from industry 2 to industry 3. Hence the change in output levels is

$$(0, -L_i/a_{i2}, L_i/a_{i3}) \tag{27}$$

Going from vertex $\alpha\gamma$ to $\alpha\delta$ does exactly the same thing. Hence the opposite edges are parallel and equal.

(ii) Going from vertex $\beta\gamma$ to $\alpha\gamma$ shifts the labor pool L_h from industry 2 to industry 1, leading to output change $(L_h/a_{h1}, -L_h/a_{h2}, 0)$. By the pattern of signs, the inner product of this vector and (27) is positive, hence the angle at $\beta\gamma$ is acute. The same argument applies to a T-face. ▒

This theorem yields an appealing comparative statics result. The physical output efficiency frontier depends on the absolute levels of $L_1, \ldots, L_m$, though the McKenzie tiling does not. What happens as $L_i \to 0$ for some particular i? All the i-labeled edges shrink in proportion, by (27). In the limit, each of the Zones 110, 011, 101 collapses to a jagged line, and Zone 111 collapses to a point; the m-country tiling has become m-1.

II.4. Price-Ratio Systems

Given a generic technology A, each face is determined by a price system (π_1, π_2, π_3) unique up to a multiplicative constant. We are interested in the price ratios, so let $P = \pi_1/\pi_2$, $Q = \pi_3/\pi_2$, and $R = \pi_1/\pi_3$. Equivalently, we can let product 2 be the numéraire, let the price system be (P, 1, Q), and define $R = P/Q$ (this second approach obscures the symmetry obtaining among P, Q and R). We are interested in the relations among the P′s, Q′s and R′s of adjoining faces.

Consider Figure 14. It reads as follows. Crossing a 1-2 edge in the direction of the arrow, and comparing P, Q, R for the face at the tail to the face at the head of the arrow, P is equal for both faces, Q rises and R falls. The results for crossing a 2-3 and a 1-3 edge are also given.

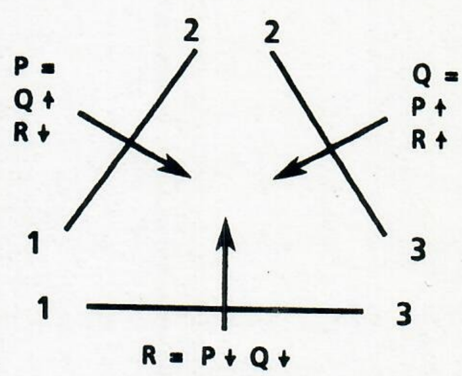

Figure 14

Theorem 19. Figure 14 is correct for any McKenzie tiling arising from a generic technology A.

Proof. We shall prove this for the 1-2 edge: the other two cases are similar. Let π', π'' be price systems for the face at the tail, head of the arrow, respectively. The 1-2 edge, labeled by row i, is common to both faces, hence $s_{i1} = 1 = s_{i2}$ in the patterns for both faces. We then have $\pi_1'/a_{i1} = \pi_2'/a_{i2}$ and also $\pi_1''/a_{i1} = \pi_2''/a_{i2}$. Hence $P' = \pi_1'/\pi_2' = \pi_1''/\pi_2'' = P''$, so the P's are equal.

The two price systems may now be taken as (P, 1, Q') and (P, 1, Q") respectively. Comparing the column signatures of the two faces we see that $c_3' < c_3''$ for the third components. This implies that $Q' < Q''$. Finally, $R' = P'/Q' > P''/Q'' = R''$. ▒

Thus, crossing any edge yields one equality and two inequality conditions. (Note that, given the equality, the two inequalities imply each other.)

The large scale structure of McKenzie tilings now comes into play. Suppose a T-face has price ratios P, Q, R assigned. Then (see Figures 12 and 13), the P propagates across the 1-2 edges, so that all faces in Zone 110 get assigned the same P. Similarly, all faces in Zone 011 get assigned the same Q, and all faces in Zone 101 get assigned the same R. Now suppose all m of the T-faces get assigned – say P_i, Q_i, R_i to the T-face of row i. Suppose Figure 9 is a U-face in the tiling; its price-ratios P, Q, R are then given by: $P = P_h$, $Q = Q_i$, $R = P_h/Q_i$. For, this face is in Zone 110 of the row h structure, and in Zone 011 of the row i structure. Similarly, any L-face will have $P = P_h$, $R = R_i$, $Q = P_h/R_i$, where its 1-2 edges are labeled by row h, and its 1-3 edges by row i. Finally, any R-face will have $Q = Q_h$, $R = R_i$, $P = Q_hR_i$, where rows h, i label its 2-3 and 1-3 edges respectively.

To conclude, assigning price ratios P_i, Q_i, R_i to all triangular faces forces a unique system of P's, Q's and R's on all faces of the tiling, via the equality relations of Theorem 19. In turn, given technology A, P_i, Q_i, R_i can be read off immediately from row i: $P_i = a_{i1}/a_{i2}$, $Q_i = a_{i3}/a_{i2}$, $R_i = a_{i1}/a_{i3}$, since the prices supporting this T-face are proportional to row i.

Figure 11 gives (P,Q) pairs for the faces of the corrected McKenzie tiling of Graham's world trade model.

II.5. Equivalent Technologies

We now come to what is perhaps the deepest part of this whole subject: the classification and enumeration of alternative technologies, and their relation to tilings and price-ratio systems.

On the set of all technologies of size (m,n), we define the following relation: A is equivalent to A' if the Jones matches of A are the same as the Jones matches of A'. That is, for any distinct countries $i_1, \ldots, i_r$ and distinct industries $j_1, \ldots, j_r$, $\{(i_1, j_1), \ldots, (i_r, j_r)\}$ is a Jones match in A if and only if it is a Jones match in A'.

Examples. (i) Multiply any row or any column of A by a positive constant. The result is equivalent to A. (This amounts to a change of measurement units of labor or output.)

(ii) Let θ be any positive number and let $a'_{ij} = (a_{ij})^\theta$, all i,j. A and A' are equivalent.

(iii) Let A and A' be equivalent. Then they are also equivalent to A" given by $a''_{ij} = a_{ij}a'_{ij}$, all i, j.

These examples follow almost immediately from the inequalities defining a Jones match, which are paradigmatically of the form $a_{11}a_{22}\cdots a_{rr} \le a_{12}a_{23}\cdots a_{r-1,r}a_{r1}$.

An appealing geometric summary of examples (ii) and (iii) can be given in terms of "logarithmic technology space". Let $t_{ij} = \log a_{ij}$, and identify technology A with the mn-dimensional vector (t_{ij}). Then, the equivalence classes of technologies are convex cones (also open, in the case of generic technologies).

An exceptional technology cannot be equivalent to a generic one since the former has tied Jones matches.

We now return to the case of n = 3 industries. It is important to distinguish labeled from unlabeled McKenzie tilings. An unlabeled tiling is simply given by a mapping from the point-down to the point-up triangles subject to the restrictions discussed above. A labeled tiling is an unlabeled tiling with additional structure: the T-faces are made to correspond to the numbers (1,..., m); the T-face label extends to its edges, and thence by "parallel propagation" to all edges of the tiling. Evidently there are m! ways of labeling an unlabeled tiling.

Theorem 20. If generic A, A' are equivalent, then they have the same labeled McKenzie tiling.

Proof. A and A' have exactly the same set of viable patterns, by the fundamental theorem of comparative advantage. For, pattern S is viable if and only if every match in S is a Jones match, which comes out the same on both A, A'.

Let (c_1, c_2, c_3) be positive integers adding up to m + 2. There is a unique maximal pattern with this column signature under each technology, and it must then be the same pattern. This pattern determines the type of face, L, U, R or T. Thus each point-up triangle will be in the same face type under both A, A'. Hence they have the same (unlabeled) McKenzie tiling.

Finally, the labels are the same, since row signature (1,1..., 3,..., 1,1) with a "3" in row i matches with the same column signature, hence gives the label i to the same T-face. ▒

Theorem 21. If generic A, A' have the same labeled McKenzie tiling, then they are equivalent.

Proof. We show that a labeled McKenzie tiling determines all Jones matches. By an argument above, the tiling determines the patterns of all facets. Choose any two countries, say h and i. The pattern S having two "1"s in both rows h and i is given by the tiling. Suppose it looks like Figure 8 in rows h and i. Note that for each pair of industries there is a match in S — e.g., {(h,1), (i,3)} in Figure 8. This must be a Jones match, S being viable. It is easy to see that this argument works for any of the six possible patterns of four "1"s in rows h and i. Hence all 2-level Jones matches are determined.

Now take any three countries — say g, h, i. The same S as above has a single "1" in row g, and still looks like Figure 8 say in rows h and i. If $s_{g1} = 1$, then {(g,1), (h,2), (i,3)} is a match in S; if $s_{g2} = 1$, then {(g,2), (h,1), (i,3)} is a match in S; if $s_{g3} = 1$, then {(g,3), (h,1), (i,2)} is a match in S. In all cases, then, we get a match, which must be Jones. Hence all 3-level Jones matches are also determined. ▒

These twin theorems show that, in a sense, this is the "right" way to bundle technologies. Equivalent technologies need not, of course, have the same system of P, Q, R price ratios on their tilings.

II.6. Weak Equivalence

Theorems 20 and 21 are for labeled tilings. Are there similar results for unlabeled tilings? We would like to characterize a relationship between two technologies A and B – call it weak equivalence – such that A and B are weakly equivalent if and only if they give rise to the same unlabeled McKenzie tiling.

Let σ be a permutation on the set {1,..., m}. The same symbol applied to a matrix A of size (m,n) gives rise to another matrix by "row permutation": σA is defined by

$$(\sigma a)_{ij} = a_{\sigma i,j} \tag{27}$$

all i = 1,..., m, j = 1,..., n.

Theorem 22. Let A, B be technologies of size (m,n). If A is equivalent to B, then σA is equivalent to σB.

Proof. Let, say, {(1,1),..., (r,r)} be a Jones match in σA, so that

$$(\sigma a)_{11}\cdots(\sigma a)_{rr} \le (\sigma a)_{1,\tau 1}\cdots(\sigma a)_{r,\tau r}$$

for all permutations τ on $\{1,\ldots,r\}$. By (27) this reads

$$a_{\sigma 1,1}\cdots a_{\sigma r,r} \le a_{\sigma 1,\tau 1}\cdots a_{\sigma r,\tau r}$$

for all τ. Since A and B are equivalent, this implies

$$b_{\sigma 1,1}\cdots b_{\sigma r,r} \le b_{\sigma 1,\tau 1}\cdots b_{\sigma r,\tau r}$$

for all τ. By (27) again, this reads

$$(\sigma b)_{11}\cdots(\sigma b)_{rr} \le (\sigma b)_{1,\tau 1}\cdots(\sigma b)_{r,\tau r}$$

for all τ. Thus $\{(1,1),\ldots,(r,r)\}$ is a Jones match in σB. ▒

Theorem 23. If pattern S is viable under technology A, then σS is viable under σA.

Proof. Let $P_1,\ldots, P_n, W_1,\ldots, W_m$ be a price system supporting S under A. Define $W_1',\ldots,W_m'$ by $W_i' = W_{\sigma i}$. Then

$$W_i'(\sigma a)_{ij} = W_{\sigma i}a_{\sigma i,j} \ge P_j$$

by (4). Further, if $(\sigma s)_{ij} = 1$, then $s_{\sigma i,j} = 1$, so this inequality becomes an equality, by (5). This proves that $P_1,\ldots, P_n, W_i',\ldots, W_m'$ is a price system supporting σS under σA. ▒

As a corollary, if A is generic, then σA is generic. For, all viable S have acyclic graphs, hence all σS have acyclic graphs; apply Theorem 6.

Now return to the case of n = 3 industries. (In the following, σ^{-1} is the permutation inverse to σ.)

Theorem 24. If generic technology A gives rise to labeled McKenzie tiling M, then σA gives rise to the same (unlabeled) tiling, but with the labels transformed by the rule $i \to \sigma^{-1}i$, $i = 1, \ldots, m$.

Proof. Let face F of M correspond to maximal pattern S under A. Then σS is a maximal pattern under σA, by Theorem 23, so it must yield a face σF of its generated tiling.

Now S and σS generate the same type face, L, U, R or T. For, if S yields a T-face, it has a row with three "1"s; then so does σS, again yielding a T-face. If S yields a U-face, then it has a pattern as in Figure 8 with two "1"s in column 2 in its special rows; this pattern is preserved in σS, hence it too yields a U-face. Similarly for L- and R-faces.

Furthermore, S and σS have the same column signature. Thus F and σF have the same location and shape. This proves A and σA yield the same unlabeled McKenzie tiling.

As for labels, if S has three "1"s in row i (so its T-face gets labeled i) then σS has three "1"s in row $\sigma^{-1}i$, by (27), so its T-face, located in the same place, gets labeled $\sigma^{-1}i$. ▒

As a corollary, if σ is not the identity permutation, then A and σA are not equivalent, since they do not yield the same labeled McKenzie tiling.

Here is the picture that emerges from these results. Let E be an equivalence class of generic technologies, with typical members A, B,... . All technologies in E (and only those) give rise to the same labeled McKenzie tiling. Let σ be a permutation on {1,..., m}, not the identity. Then σE, consisting of

σA, σB,..., is a distinct equivalence class giving rise to the same tiling with permuted labels. As σ ranges over all possible permutations, we get a cluster of exactly m! equivalence classes. The union of these is the set of generic technologies giving rise to a single unlabeled tiling. These larger classes yield the weak equivalence relation sought above: A and B are weakly equivalent – i.e., are in the same united set – if and only if A is equivalent to σB for some permutation σ.

II.7. Technology-Generated Tilings

Consider the set E of all weak equivalence classes of generic technologies, and the set T of all (unlabeled) McKenzie tilings (both for m countries). Let f be the mapping sending each technology to its tiling; f has the same value for technologies in the same class, and different values for technologies in different classes, so we may think of it as an injective mapping from E to T. But what is the range of f – that is, which tilings are actually generated by some technology?

We begin this investigation with the following rather deep result, of considerable interest in its own right.

Theorem 25. Given a McKenzie tiling, and for each face given positive numbers P, Q, R with P = QR and such that, across faces, the conditions of Figure 14 are satisfied. Then there exists a generic technology that yields this tiling and has the P, Q, R's as its price-ratio system.

Proof. Step 1: First we derive some consequences of the relations in Figure 14. Label the tiling, and let the T-face labeled by i have P_i, Q_i, R_i as its assigned numbers. This face yields a zonation of the set of all faces as in Figures 12 and 13 into (at most) seven zones. Let F be <u>any</u> face and let P, Q, R be the numbers assigned to F.

$$\left.\begin{array}{lll} \text{If F is in Zone} & 111 & \text{then } (P, Q, R) = (P_i, Q_i, R_i) \\ & 110 & P = P_i,\ Q < Q_i,\ R > R_i \\ & 011 & Q = Q_i,\ P < P_i,\ R < R_i \\ & 101 & R = R_i,\ P > P_i,\ Q > Q_i \\ & 100 & P > P_i,\ R > R_i \\ & 010 & P < P_i,\ Q < Q_i \\ & 001 & Q > Q_i,\ R < R_i \end{array}\right\} \qquad (28)$$

The statement for Zone 111 is immediate; for Zone 110 it follows from Figure 14 by "parallel propagation" across 1-2 edges; for Zone 011 by propagation across 2-3 edges; and in Zone 101 by propagation across 1-3 edges. The other three cases are a little more difficult. Suppose, for example, that F is in Zone 100 and we want to show that $R > R_i$. No matter which type face F is (L, U, R or T) at least one of the two following situations obtains: F  or F  - i.e., F has a 1-2 or a 2-3 edge which can be crossed as shown, F lying at the tail of the arrow. Each of these crossings leads to a lower R. If this new face is still in Zone 100, repeat the operation. Eventually we get to a face in Zone 110 or 101. The R-value of <u>this</u> face is $\geq R_i$, by the results above for these zones. Putting all of these inequalities together yields $R > R_i$.

Similarly, let F again be in Zone 100; to show that $P > P_i$ consider the two following situations:  or , a 2-3 or 1-3 edge with F at the tail of the arrow. At least one of these obtains for any face type, and each crossing leads to a lower P. Iterate as before until a face in Zone 110 or 101 is reached. The P-value of <u>this</u> face is $\geq P_i$, by the results above. Putting these inequalities together yields $P > P_i$.

The procedure for F in Zone 010 or 001 is similar. (In each case, from Figure 14 there are exactly two combinations of edge orientation, 1-2, 2-3, or 1-3, and direction of crossing that change P, Q, or R in the "right" direction as dictated by (28). Each of L, U, R, T can utilize one of these combinations. Iteration leads from Zone 010 eventually to Zone 110 or 011, and from Zone 001 to Zone 101 or 011, supplying the last leg of the argument as above). This proves (28).

Step 2: Define technology A as follows: $a_{i1} = P_i$, $a_{i2} = 1$, $a_{i3} = Q_i$. Note that the middle column is all "1"s. P_i and Q_i are the numbers assigned to the T-face labeled i, which we now call T_i. We will show that A satisfies the theorem. First we prove that A is in general position.

Pick any two rows, say h, i, and suppose that F, the face with edges labeled h, i, is of type U as in Figure 9. (A similar argument works if F is of type L or R.) Then F is in Zone 110 of T_h and also in Zone 011 of T_i (Figure 15). It then follows from (28) that the following relations obtain:

$$P_i > P = P_h$$
$$Q_h > Q = Q_i \qquad (29)$$
$$R_i > R > R_h$$

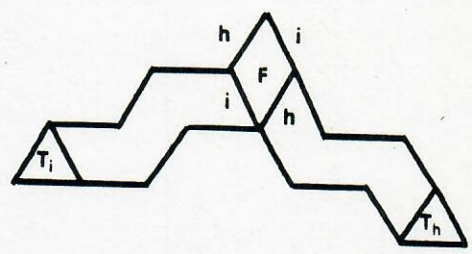

Figure 15

where P, Q, R are the numbers assigned to face F. The fact that $P_h \neq P_i$, $Q_h \neq Q_i$ and $R_h \neq R_i$ shows there are no tied Jones matches at the 2-level.

For the 3-level, take rows g, h, i and let F again be as above. (Again, similar arguments work for F of type L or R.) Let S be the pattern of F. S is determined by the tiling, and the zoning of F shows that S looks like Figure 8 in rows h and i. The location of the single "1" in row g of S gives rise to three cases.

Case (i). $s_{g1} = 1$. {(g,1), (h,2), (i,3)} is a match in S, and we now show that it is the unique Jones match. Its score $a_{g1}a_{h2}a_{i3} = P_gQ_i$, which we must show is less than the scores of the other five matches: P_gQ_h, P_iQ_g, P_hQ_i, P_iQ_h and P_hQ_g. This is readily done by using (29) together with the two additional facts: $P > P_g$, $R > R_g$ (these follow from (28) on noting that F is in Zone 100 of T_g). For example, $R_gQ_i < RQ = P = P_h$, which yields $P_gQ_i < P_hQ_g$. The other inequalities are left as exercises.

Case (ii). $s_{g2} = 1$. Show {(g,2), (h,1), (i,3)}, a match in S, is the unique Jones match by utilizing (29) and the additional facts: $P < P_g$, $Q < Q_g$. (These follow from F now being in Zone 010 of T_g.)

Case (iii). $s_{g3} = 1$. Show {(g,3), (h,1), (i,2)}, a match in S, is the unique Jones match by using (29) and the additional facts: $Q > Q_g$, $R < R_g$, which follow from F being in Zone 001 of T_g. Details are omitted in cases (ii) and (iii).

This proves A is in general position.

Step 3: Let F be any face of the tiling, with column signature (c_1, c_2, c_3) (adding up to m + 2). Let S be the pattern of F. We will show that S is viable under A. Since S has m + 2 "1"s it is a maximal pattern under generic A, hence yields a face of the tiling generated by A. Thus this tiling coincides with the original.

Let (P, Q, R) be the numbers assigned to F. We show that the price system (P, 1, Q) strictly sustains S under technology A. Do this row by row. In row i we are to show that $s_{ij} = 1$ if and only if the j-th term of $(P/a_{i1}, 1/a_{i2}, Q/a_{i3}) = (P/P_i, 1, Q/Q_i)$ is maximal, j = 1, 2, 3. Now, if F is in Zone xyz of T_i, then the i-th row of S <u>is</u> xyz. Hence we need show only that the subset of maximal elements in $(P/P_i, 1, Q/Q_i)$ matches the zoning code exactly. This follows readily from (28):

In Zone 111, the triple is (1,1,1). In Zone 110 it is $(1, 1, Q/Q_i)$ where $Q < Q_i$. In Zone 011 it is $(P/P_i, 1, 1)$, with $P < P_i$. In Zone 101, $R = R_i$, so $P/P_i = Q/Q_i$, these both being > 1. In Zone 100, $P/P_i > 1$, and also $> Q/Q_i$ since $R > R_i$. In

Zone 010, P/P_i and Q/Q_i are both < 1. Finally, in Zone 001, $Q/Q_i > 1$, and also $> P/P_i$, since $R < R_i$. Thus S is viable under A, and A yields the original tiling.

Step 4: Finally, A also yields the (P, Q, R)'s as its price-ratio system. For, on face T_i, prices (π_1, π_2, π_3) are proportional to row i, $(P_i, 1, Q_i)$. The price ratios are then $(P_i/1, Q_i/1, P_i/Q_i) = (P_i, Q_i, R_i)$, the original assignments on T-faces, and this determines all other assignments uniquely by "parallel propagation". ▒

This theorem can be rephrased by combining it with its much easier converse (Theorem 19) to read:

Theorem 26. A McKenzie tiling is generated by some technology in general position if and only if there exists a system of positive numbers (P, Q, R) assigned to each of its faces satisfying the conditions of Figure 14, and P = QR.

Here is an application. Suppose one is given an arbitrary McKenzie tiling, and wants to know whether it arises from some generic technology. The easiest way is to look for a price-ratio system. Taking logarithms preserves the relations of Figure 14 for each edge, and also yields log P = log Q + log R for each face. Thus we get a system of <u>linear</u> relations, which can be tested and solved by linear programming (Gale, pp. 44-49, 121). If a solution exists, it immediately yields a generating technology.

It is much easier to work with price-ratio systems than directly with technologies. We now apply these ideas to show that a large number of tilings <u>are</u> generated by technologies — namely, all those with <u>at</u> <u>most</u> <u>five</u> countries.

Theorem 27 below is not too interesting in itself, but very useful. Some preliminaries follow. The <u>bottom</u> <u>tier</u> of a McKenzie tiling consists of the faces having column signatures $(c_1, 1, c_3)$. Consider the <u>subtiling</u> formed by discarding all of these faces, and also discarding the bottom halves of any U-faces with column signatures $(c_1, 2, c_3)$, converting their point-up com-ponents into T-faces (Figure 16). The result is a new McKenzie tiling of size m-1.

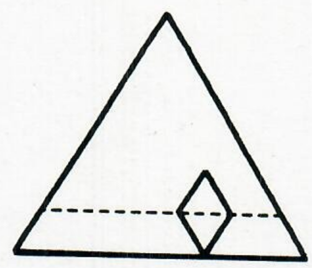

Figure 16

Suppose this subtiling had a price-ratio system (P, Q, R) on its faces. Then, we will show that the original tiling also has a price-ratio system, <u>if</u> the bottom tier has one of the following five forms (see Figure 17).

(17a) Just one T-face.

(17b) Just two T-faces, in

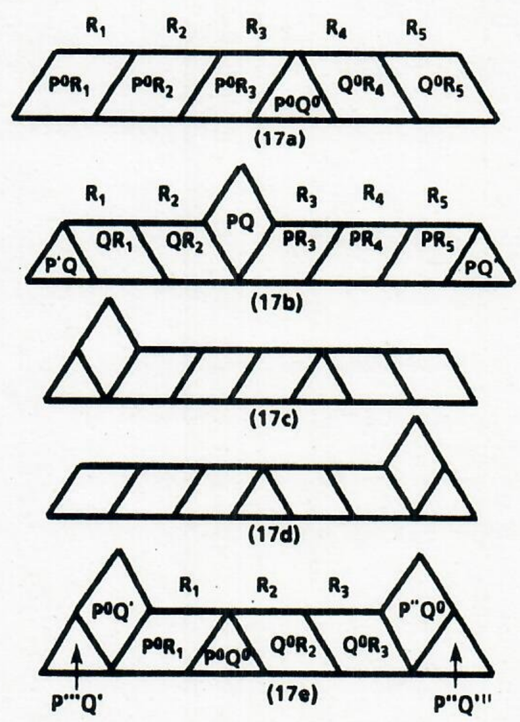

Figure 17

the corners. (There is then exactly one U-face poking up into the next tier; it can be anywhere).

(17c,d) Just two T-faces, one in a corner, with a U-face next to it.

(17e) Just three T-faces, two in the corners with U-faces next to them.

Theorem 27. If the subtiling has a price-ratio system, and the bottom tier has any of these five forms, then the original McKenzie tiling has a price-ratio system.

Proof. (17a) All faces in the subtiling keep their assigned (P, Q, R)'s. Let $R_1, \ldots, R_5$ (say) be the R-values assigned in order to the bottom tier of the subtiling. Then $R_1 > \ldots > R_5$ (since R must decrease going from left to right by Figure 14.) Suppose, say, that the T-face has its apex between the R_3 and R_4 faces (Figure 17a).

Assign P^o, Q^o to the T-face to satisfy the following conditions: P^o exceeds every P-value in the subtiling, Q^o exceeds every Q-value in the subtiling, and $R_3 > (P^o/Q^o) > R_4$. (Clearly these conditions can be met.) The equality relations across edges then force the complete assignment shown in Figure 17a, and it is easy to check that all inequality conditions across edges are also satisfied.

(17b) Let the lopped-off U-face (which is a T-face in the subtiling) have (P, Q) assigned to it, and let, say, $R_1, \ldots, R_5$ be the R-values assigned in order to the other faces of the subtiling's bottom tier (Figure 17b). We then have $R_1 > R_2 > (P/Q) > R_3 > R_4 > R_5$. All faces in the subtiling keep their assigned price ratios, with the restored U-face inheriting the (P, Q) assigned to its truncation. Equality relations then force the assignments shown in 17b, except for P', Q' in the corners. For these choose $P' > QR_1$ and $Q' > P/R_5$. It is readily checked that all inequality conditions across edges are now satisfied.

(17c, d, e) We will just do (17e). ((17c) and (17d) are similar but easier and left as exercises.) Call the two U-faces U', U", and their lopped-off versions T', T", carrying price-ratios (P', Q'), (P", Q"), respectively. Let the other faces in the subtiling bottom tier have assignments, say, R_1, R_2, R_3. Then $(P'/Q') > R_1 > R_2 > R_3 > (P''/Q'')$. Suppose, say, that the middle T-face has its apex between the R_1- and R_2-faces. Assign price-ratios (P^o, Q^o) to this face to satisfy these conditions: P^o, Q^o exceed the values of all P's, Q's in the subtiling, respectively, and $R_1 > (P^o/Q^o) > R_2$.

The subtiling is to retain all its price-ratio values, but U' is assigned (P^o, Q') (not (P', Q')) and U" is assigned (P'', Q^o) (not (P", Q")). Equality conditions across edges then force all other price-ratio assignments except for P"', Q"'in the corners (Figure 17e). For these choose $P''' > P^o$, $Q''' > Q^o$. It is readily checked that all inequality conditions across edges are now satisfied. ▒

Theorem 27 has been stated in terms of the bottom tier, along the 1-3 border. But by symmetry an identical result holds for the tier along the 1-2 border (all faces with column signatures (c_1, c_2, 1)) and also for the tier along the 2-3 border (all faces with signatures (1, c_2, c_3)). This triple-strength theorem is the one used in the following proof.

Theorem 28. For any McKenzie tiling of size m = 1, 2, 3, 4 or 5, there is some technology in general position that yields it.

Proof. We need show only that such a tiling has a price-ratio system. For m = 1 this is trivial. Now proceed by induction. It suffices to show how to go from m = 4 to m = 5, since the same arguments, only simpler, get us from m = 1 to 2, from 2 to 3, and from 3 to 4.

Suppose, then, that every McKenzie tiling of size m = 4 has a price-ratio system, and consider a tiling of size m = 5. This has five T-faces.

Case (i). No T-face is in a corner. Of the three outer tiers (along border 1-2, 2-3 and 1-3 of the quincunx) at least one has only a single T-face. Hence we can apply Part (a) of Theorem 27.

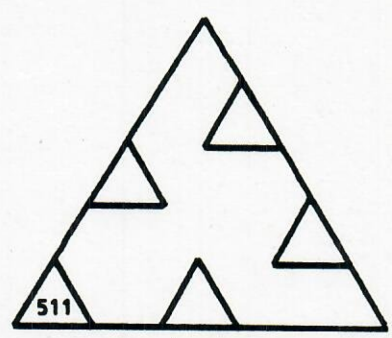

Figure 18

Case (ii). Exactly one T-face is in a corner – say at signature 511, without loss of generality. If some outer tier has just one

T-face, apply Part (a) of Theorem 27. If every outer tier has at least two T-faces, they must be distributed as in Figure 18. Consider the face adjoining 511. It is either type U or type R. If it is type U, apply Part (c) of Theorem 27 to the bottom (1-3 border) tier. If it is type R, apply Part (d) of Theorem 27 to the 1-2 border tier.

Case (iii). Exactly two T-faces are in corners – say at 511 and 115. If there is no other T-face in the bottom tier, apply Part (b) of Theorem 27 to it. If there _is_ one, and if no outer tier has a single T-face, they must be distributed as in Figure 19.

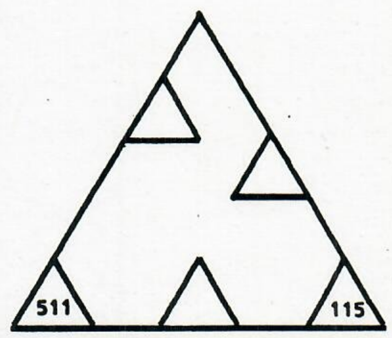

Figure 19

Let F', F" be the faces adjoining 511, 115, respectively. If F' is type R, proceed as in case (ii). If F" is type L, apply Part (c) of Theorem 27 to the outer tier on the 2-3 border. If neither of these holds, then both F', F" are of type U. Then apply Part (e) of Theorem 27 to the bottom tier.

Case (iv). All three corners are T-faces. Then some outer tier has no other T-face. Apply Part (b) of Theorem 27 to it. ▒

This theorem covers a lot of ground, for there are 3135 distinct (unlabeled) McKenzie tilings of size 5 (and 120·3135 = 376,200 labeled tilings) – hence there are this many technology classes with 5 countries.

For all we know to this point, any McKenzie tiling might be yielded by some generic technology. But this is not so.

Theorem 29. The McKenzie tiling in Figure 20 cannot arise from any generic technology.

Proof. We show that this tiling has no price-ratio system. The T-faces are labeled 1,..., 7. Let (P_i, Q_i, R_i) be the price-ratios for the i-th T-face. The equality constraints of Figure 14 then propagate the ratios to force a unique system on the entire tiling. In Figure 20 we have listed only those entries used in the following argument. The seven edges stressed in Figure 20 give rise to seven inequalities:

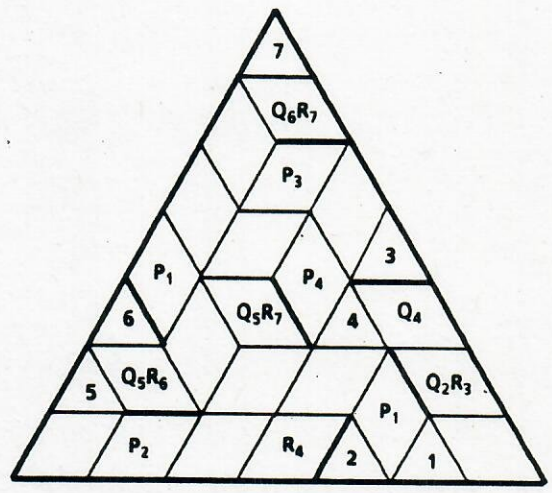

Figure 20

$$\left.\begin{array}{l} P_3 > Q_6R_7 \\ Q_4 > Q_3 \\ Q_5R_7 > P_4 \\ P_6 > P_1 \\ P_2 > R_6Q_5 \\ R_4 > R_2 \\ P_1 > Q_2R_3 \end{array}\right\} \quad (30)$$

Multiplying these inequalities together and cancelling P_i against Q_iR_i yields the contradiction 1 > 1! ▒

Thus a counterexample appears at the 7-country level — and also at every higher level, since we can let Figure 20 be the first seven tiers of a larger tiling. We conjecture that as m rises, a vanishingly small proportion of McKenzie tilings are actually generated by a technology. The problem of characterizing these tilings is still open.

It is also not known whether a counterexample exists at the 6-country level. (We conjecture it does not). Since there are 81,462 unlabeled McKenzie tilings at this level (and 720·81462 = 58,652,640 labeled tilings), resolution either way will not be easy.

There still remain some generalities holding for all sizes m. Start with the following preliminary result.

Theorem 30. In a McKenzie tiling of size m, let $|L|$, $|U|$, $|R|$ be the number of L-, U-, R-faces present, respectively, and let (c_{i1}, c_{i2}, c_{i3}) be the column signature of the i-th T-face. Then

$$|L| + \Sigma_i c_{i1} = |U| + \Sigma_i c_{i2} = |R| + \Sigma_i c_{i3} = (m^2 + m)/2 \qquad (31)$$

Proof. For the i-th T-face, the number of faces in Zone 101 is $c_{i2} - 1$. To see this, start counting from the bottom tier (the first face has a column signature of the form (a, 1, b)) and note that upcrossing a 1-3 edge (—↑—) increases the middle signature component by one. The count continues up to T_i with component

c_{i2}. Similarly, the number of faces in Zones 110, 011 is $c_{i3} - 1$, $c_{i1} - 1$, respectively.

On the other hand, the Zones 110 for distinct T_i's are disjoint, and each U-face appears exactly once in the union of these zones; also, it appears exactly once in the union of the Zones 011 (see Figure 15). Similarly, each L-face appears once in the union of Zones 110 and once in the union of Zones 101. Finally, each R-face appears once in the union of Zones 101 and once in the union of Zones 011.

These arguments yield three equations

$$|U| + |R| = \Sigma_i(c_{i1} - 1)$$
$$|L| + |R| = \Sigma_i(c_{i2} - 1) \qquad (32)$$
$$|L| + |U| = \Sigma_i(c_{i3} - 1)$$

But

$$|L| + |U| + |R| = (m^2 - m)/2, \qquad (33)$$

the number of quadrilateral faces. Subtracting each of the equations (32) in turn from (33) yields (31). ▒

For example, in Figure 20, $|L| = 8$, $|U| = 7$, $|R| = 6$, $\Sigma_i c_{i1} = 20$, $\Sigma_i c_{i2} = 21$, $\Sigma_i c_{i3} = 22$.

Theorem 31. For any size m, a McKenzie tiling with any of the following properties has all of them (see Figure 21).

(i) All quadrilateral faces of type U.

(ii) All T-faces on the bottom tier.

(iii) Generated by a technology A of the following form:

$a_{11} < a_{21} < \ldots < a_{m1}$,

$a_{13} > a_{23} > \ldots > a_{m3}$, (34)

$a_{i2} = 1$, all i.

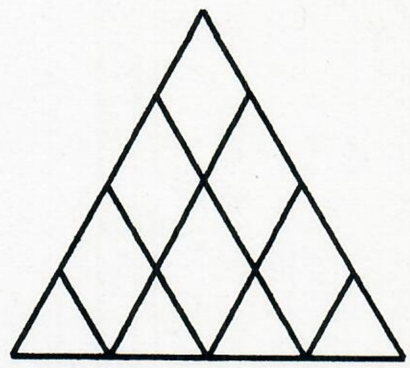

Figure 21

Proof. (i) <u>if and only if</u> (ii): By (31), $|U| = (m^2 - m)/2$ if and only if $\Sigma_i c_{i2} = m$, which holds if and only if $c_{i2} = 1$, all i.

(iii) <u>implies</u> (ii): Any technology of the form (34) is in general position ({(g,1), (h,2), (i,3)} is a Jones match if and only if g < h < i; 2-level uniqueness is obvious.)

Let face T_i have pattern S. $(a_{i1}, 1, a_{i3})$ is a price system for S. If h < i, then $a_{i1}/a_{h1} > 1$, so $s_{h2} = 0$; if h > i, then $a_{i3}/a_{h3} > 1$, so again $s_{h2} = 0$. Thus the middle column of S has only one "1": $c_{i2} = 1$.

(i) <u>and</u> (ii) <u>imply</u> (iii): Since technology (34) is in general position, it yields <u>some</u> McKenzie tiling, which by the above satisfies (ii). But there is only one tiling satisfying (ii). ▒

All technologies satisfying (34) are equivalent, by the remarks in the proof concerning Jones matches. They say in a sense that industry 2 is always "between" the other two industries.

Theorem 31 singles out industry 2. Needless to say, there is a similar result for industry 1 (all quadrilaterals L-type) and for industry 3 (all R-type). Correspondingly, rotate Figure 21 by 120° and in (34) permute the columns. These three technology classes are basically identical if we ignore column labels, and may be considered the simplest possible class. There is an important generalization to the m-by-n case and beyond, under the term "simple structure" (see below).

Now we go to the opposite extreme:

Theorem 32. For any size m, a McKenzie tiling with any of the following properties has all of them (see Figure 22).

(i) No quadrilateral faces of type U.

(ii) Each horizontal tier contains exactly one T-face.

(iii) Generated by a technology A (in general position) of the following form:

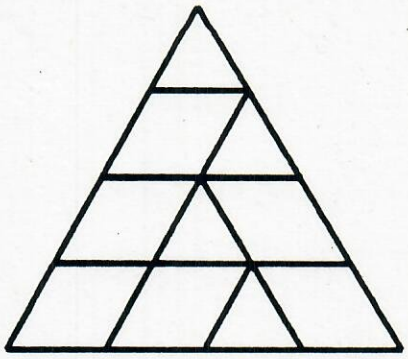

Figure 22

$$a_{11} < a_{21} <\ldots< a_{m1},\ a_{13} < a_{23} <\ldots< a_{m3},\ a_{i2} = 1, \text{ all } i. \qquad (35)$$

Proof. (ii) <u>implies</u> (i): For column signatures of T-faces, the set $\{c_{12}, \ldots, c_{m2}\} = \{1, \ldots, m\}$, hence adds up to $(m^2 + m)/2$. Hence, by (31), $|U| = 0$.

(iii) implies (i): Let F be a quadrilateral face with pattern S having two "1"s apiece in rows h, i, where h < i. Suppose $s_{h2} = 1$. Then $s_{i1} = 0 = s_{i3}$, else comparative advantage is violated. This contradiction shows $s_{h2} = 0$, so F cannot be type U.

(i) implies (ii) and (iii): By induction on m. Without U-faces, the bottom tier must be of the form of Figure 17a, completing the induction for (ii). Further, as in the proof of Theorem 27, Part (a), a price-ratio system extends to an extra tier of this sort by choosing P^o, Q^o larger than all other P, Q's in the subtiling. This extended price-ratio system translates to a technology with an extra m-th row of the form $(P^o, 1, Q^o)$. Thus (35) remains satisfied, completing the induction for (iii). ▒

Note that this includes as special cases the "rotated" versions above where all quadrilaterals are R-faces or all are L-faces. Here too, rotating Figure 22 by 120° gives versions with no R-faces, or with no L-faces.

One implication of Theorem 32 is that, if a tiling does not arise from any generic technology (as in Figure 20) it must contain some of all three types of faces, L, U and R.

The following is a fairly general existence theorem applying to any size m.

Theorem 33. Let $|L|$, $|U|$, $|R|$ be three natural numbers adding up to $(m^2 - m)/2$. Then there exists a McKenzie tiling of size m arising from a generic technology, and having these numbers of L-, U-, and R-faces present, respectively.

Proof. By induction on m. Trivial for $m = 1$. Assume the statement holds for m, and let $|L|$, $|U|$, $|R|$ add up to $((m + 1)^2 - (m + 1))/2 = (m^2 + m)/2$.

Consider the three conditions: $|L| + |R| \leq m-1$, $|L| + |U| \leq m-1$, $|R| + |U| \leq m-1$. If all of these are true, then

$$3m - 3 \geq 2(|L| + |U| + |R|) = m^2 + m$$

which cannot be. Hence at least one of these conditions is false, so say $|L| + |R| \geq m$. (The argument in the other two cases is similar.) Then there exists an integer λ satisfying

$$\text{Max}\left[0, m - |R|\right] \leq \lambda \leq \text{Min}\left[|L|, m\right].$$

Consider the triple

$$\left[|L| - \lambda, |U|, |R| - m + \lambda\right]. \qquad (36)$$

These are natural numbers adding up to $|L| + |U| + |R| - m = (m^2 - m)/2$. By induction hypothesis there is a McKenzie tiling M of size m having a price-ratio system, and having (36) as its number of L-, U-, R-faces, respectively.

Now tack on an extra tier having one T-face, λ L-faces, and m - λ R-faces (Figure 23). The result is a tiling of size m + 1 having exactly |L|, |U|, |R|, L-, U-, R-faces, respectively. Further, by Theorem 27, Part (a), the price-ratio system extends to the new tiling, hence it arises from a generic technology. This completes the induction. ▒

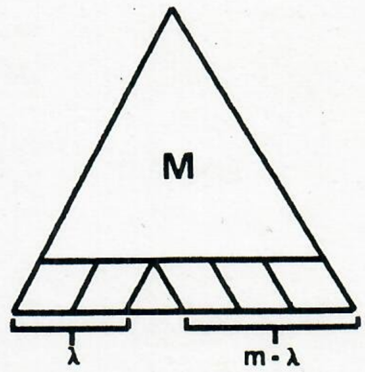

Figure 23

II.8. Admissible Designations

The placement of T-faces in tilings plays an important role in several of the developments above: in large-scale structure and number of L-, U-, R-faces. The following investigates this placement directly.

In a size-m quincunx (Figure 6 for m = 4) there are $(m^2 + m)/2$ point-up triangles, and m of these are T-faces in a McKenzie tiling. But this subset of m cannot be chosen arbitrarily. For example, in Figure 24 it is impossible for all three of the point-ups to be T-faces, since the point-down in the middle has to amalgamate with one of them.

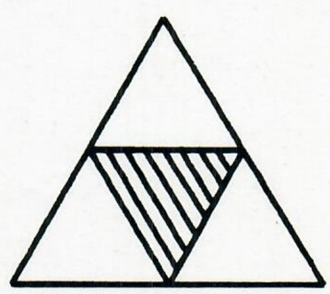
Figure 24

An r-subquincunx is a part of the original quincunx that is itself a quincunx of size r. (Formally it is defined by its three corners, which have signatures (a, b-r, c-r), (a-r, b, c-r), and (a-r, b-r, c), for some integers a, b, c $\geq$ r adding up to m + 2r. r = 1 is a single point-up triangle, r = m the entire original; Figure 24 has r = 2.)

A T-designation is simply a subset of size m of the set of point-up triangles. A T-designation is admissible if, in every r-subquincunx, at most r of the point-ups are designated.

Theorem 34. For a given T-designation, there exists a McKenzie tiling having its T-faces at that designation if and only if the T-designation is admissible.

Partial Proof. Only if. An r-subquincunx has $(r^2 + r)/2$ point-up triangles and $(r^2 - r)/2$ point-down triangles. To form a McKenzie tiling, the latter must map injectively into the former. This leaves only $(r^2 + r)/2 - (r^2 - r)/2 = r$ point-ups to spare. ▒

(The proof of the converse, that admissibility guarantees the existence of a tiling with those T-faces, is a complicated nested induction argument, and is omitted.)

In general, an admissible T-designation does not determine a McKenzie tiling uniquely. For example, if either configuration in Figure 25 occurs in a tiling, it can be transformed into the other without disturbing any T-faces. In the counterexample of Figure 20, this configuration occurs three times. At two of these times, the transformation of Figure 25 reverses an inequality in (30), and therefore disrupts the delicate proof of Theorem 29. These considerations suggest the following.

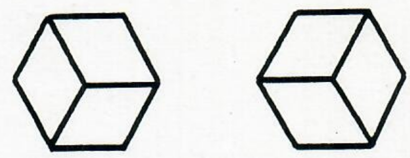

Figure 25

Conjecture. For any size m, for any admissible T-designation, there exists a McKenzie tiling having its T-faces at that designation, <u>and</u> arising from a generic technology.

Note that a T-designation <u>does</u> determine the number of L-, U- and R-faces in a compatible tiling, via (31). Hence the conjecture is a strengthening of Theorem 33.

II.9. Enumeration of Technology Types

The following table gives the number of distinct unlabeled McKenzie tilings, and the number of admissible T-designations for various values of m, the number of countries.

Table 1

Countries	Unlabeled Tilings	Admissible Designations
1	1	1
2	3	3
3	18	17
4	187	150
5	3135	1848
6	81462	?

To get the number of labeled McKenzie tilings, multiply by m!. Up through m = 5 (and probably m = 6) this is also the number of distinct equivalence classes of technologies in general position, which provides the most natural way of classifying technology types. (Classification by the location of T-faces may also be useful, hence the last column.)

A word on how these numbers were calculated. For T-designations, it is easiest to enumerate the number of <u>in</u>admissible ones and subtract from $\binom{(m^2+m)/2}{m}$, the total number of designations. Count the number violating admissibility at the 2-level, the 3-level, etc.

For tilings, consider first m = 3 countries. There are three point-down triangles, and each can map to three neighbors, yielding 27 mappings. Of these, 9 are not injective, leaving 18. This kind of "inclusion-exclusion" argument is not profitable for larger m. Instead proceed inductively as follows. For m, calculate the number of unlabeled tilings having each possible configuration of T-faces in its bottom tier (there are $2^m - 1$ cases). Now go to m + 1 with a specified configuration of k T-faces in <u>its</u> bottom tier. These must intersperse with k-1 U-faces (cf. Figures 17b, c, d, and e). Now lop off this bottom tier, leaving a size m tiling, with the top halves of the U-faces among the T-faces of <u>its</u> bottom tier. This allows enumeration based on the m-level data. (Details are omitted. This algorithm is not too satisfactory since the amount of work doubles with each successive m. Nonetheless the table was hand-calculated using it).

For m > 6 the enumeration of tilings becomes less interesting in view of counterexamples such as Figure 20. The problem of enumerating technology classes here remains open.

II.10. The Input Efficiency Frontier for Three Countries

We have been focusing almost exclusively on the m-country, 3-industry case. The output efficiency frontier is embedded in 3-dimensional space, and is in general simpler than the input frontier. For the 3-country, n-industry case it is the input efficiency frontier that is embedded in the 3-space (L_1, L_2, L_3) of resource inputs. We may again represent this frontier by McKenzie tilings. In fact, the entire theory is virtually identical to the one developed above. This arises from the following simple observation.

Theorem 35. If pattern S is viable under technology A (of size (m, n)), then the transpose of S is viable under the transpose of A.

Proof. Let $P_1, \ldots, P_n$, $W_1, \ldots, W_m$ be a price system supporting S under A. Then $P'_1, \ldots, P'_m$, $W'_1, \ldots, W'_n$ is a price system supporting S-transpose under A-transpose, where $P'_i = 1/W_i$, $W'_j = 1/P_j$, i = 1,..., m, j = 1,..., n. This follows immediately from (4) and (5). ▒

Thus the viable patterns under 3-by-n technology A are exactly the transposes of the viable patterns under the n-by-3 technology A-transpose. The McKenzie tiling in input space determined by the former is then identical to the McKenzie tiling in output space determined by the latter; from this flows everything else. (It is a remarkable fact that, physically, the output efficiency frontier is "concave to the origin" while the input efficiency frontier is "convex to the origin" — yet this makes no difference to the ensuing theory.)

There is one instructive contrast concerning the price-ratio system. For an input McKenzie tiling arising from a generic technology, each face is supported by a wage triple (W_1, W_2, W_3) rather than a price triple (π_1, π_2, π_3): Given industry j, $s_{ij} = 1$ for any country i minimizing $W_i a_{ij}$. The P, Q, R′s then are wage-ratios rather than price-ratios. Because the patterns S satisfy a minimizing condition (rather than maximizing π_j/a_{ij}) the inequality results of Theorem 19 (Figure 14) are all <u>reversed</u>.

This makes no real difference to any subsequent results, except that inequality signs must be reversed at appropriate places. For example, the counterexample of Figure 20 still works: Inequalities (30) are all reversed, still yielding the same contradiction.

III. Conclusions: Open Problems, Further Work and Applications

This concludes our exposition. The extension of the theory of comparative advantage to many countries and industries yields a surprisingly rich theory.

We now present an agenda for further work. The "internal" agenda involves filing in the gaps, testing conjectures and solving open problems. The "external" agenda involves connecting these ideas with the rest of economic theory.

The "accounting" or "analytic" portion of our paper worked with the general m-country, n-industry situation, but the "synthetic" portion worked almost exclusively with the 3-industry case (or 3-country, by transposition). There are, of course, some fairly deep unsolved problems even in this case; but the main challenge is to extend the development to the case of n = 4 or more industries.

As a beginning, consider the possible row signatures for maximal patterns under generic technology for n = 4. These are of three types: (1, 1,..., 4,..., 1, 1) with a single "4", (1, 1,..., 2,..., 3,..., 1, 1) with a "2" and a "3", and (1, 1,..., 2,..., 2,..., 2,..., 1, 1) with three "2"s. Row signatures determine the internal structure of their corresponding "faces" (which here are 3-dimensional polyhedra), and in fact, these types yield tetrahedra, triangular prisms, and parallelepipids, respectively, as one easily finds. The numbers of these faces are also known for any generic technology by Theorem 15, being m, $m^2 - m$, and $\binom{m}{3}$, respectively, for m countries. But how these pieces fit together, and in how many ways, is not known. What is the 4-industry analog of McKenzie tilings?

There is little question this program can be carried out, using the methods of this paper. Geometric intuition begins to fail for larger n, and more reliance must be placed on the column signatures (the "addresses") of the facets involved, so the theory will be more algebraic in tone. Here is a concrete challenge: Find the number of distinct technology equivalence classes for the case of 4 countries and 4 industries.

A second major task is to extend the synthetic portion to exceptional technologies. This is essentially terra incognita despite some preliminary observations by McKenzie (p. 175, note 2). There are both mathematical and economic reasons for investigating them. First, in technology space, the exceptionals form the borderlines between equivalence classes of generic technologies. (Moving from one class to another, some Jones match must switch. At the point of crossing, at least two matches must tie for lowest score, so the technology is exceptional.) Learning the facet structure for these technologies should provide insight into the relations among "adjacent" McKenzie tilings.

Further, a number of our results should hold for all technologies, not just generic ones. It would be surprising indeed if the tiling of Figure 20 could arise from <u>any</u> technology. The trouble is that the simple relations involving row and column signatures and viable patterns break down for exceptional technologies, so the entire quincunx-tiling approach is undermined.

As for the economic motives for studying exceptional technologies, consider the 2-by-2 case (Figure 1). It is not unusual in the real world for both countries to produce both goods, implying ad = bc. This occurs only if, contrary to the Ricardian assumption, there is some "neoclassical" flexibility in the technical coefficients a_{ij}. In this case equilibration <u>may</u> stop short of complete specialization by either country, and get hung up at an "exceptional" structure. (Interpret the a_{ij}'s here as being marginal costs). It is still true, of course, that this technology can be approximated by another in general position with arbitrarily small error. But the <u>qualitative</u> patterns change discontinuously, so the exceptionals deserve separate study.

(For the m-by-2 or 2-by-n cases, exceptional technologies are easy to deal with: they simply reduce the effective number of countries or industries below m or n, respectively. For m, n $\geq$ 3 we are not so lucky.)

Still another situation of real world importance is where some countries cannot produce at all in some industries ($a_{ij} = \infty$, so $s_{ij} = 0$). To see that our work does not cover this case, note that for a_{ij} finite, no matter how large, pattern S, with $s_{ij} = 1$ and zeros elsewhere, is viable. Extending this to a maximal pattern, we conclude that for any country i and any industry j, there is some face under which i produces in j. Clearly this face must disappear for $a_{ij} = \infty$, so the McKenzie tiling structure — or its higher-dimensional analog — breaks down.

We now turn to the wider economic implications. The theory of comparative advantage is a fragment of a more general model, since in itself it says nothing about demand conditions, not to speak of dynamical changes in technology, resource supplies, etc.

The output efficiency frontier may be thought of as representing a set-valued supply function (a correspondence): When confronted with the price system $P = (P_1, \ldots, P_n)$ it responds by choosing any point $(X_1, \ldots, X_n)$ of the facet F_p. This output vector is attained by a pattern of specialization and division of labor that is the main focus of attention of the theory.

The historical association of the theory of comparative advantage with international trade is misleading in two respects. First, it is a theory of who produces what, not who exports what. The latter requires additional information concerning demand, the former does not. (The particular point chosen on the efficiency frontier also requires demand information, of course, but it is the frontier as a unit and the associated configuration of patterns that is the object of the theory, and <u>this</u> does not depend on demand).

The association with international trade theory may also obscure the scope of the principle of comparative advantage. It is <u>the</u> explanation for the pattern of specialization and division of "labor" at any level: which parcels of land get assigned which uses, which occupations go into which industries, which workers enter which occupations, all the way down to the interpersonal division of labor – why Adam delved and Eve span. This broad scope is useful in the theory of international trade itself. By applying it to the problem of assigning factors to industries, Roy J. Ruffin has given a "Ricardian" underpinning to Heckscher-Ohlin theory (Ruffin, 1988).

Here is another approach that yields similar conclusions. Suppose each country has a convex production possibility set (so that its output efficiency frontier is "concave to the origin".) For n = 3 industries, it seems likely that this frontier may be approximated as closely as desired by a "physical" McKenzie tiling, composed of parallelograms and triangles, which arises from some technology. If so, then a country acts <u>as if</u> it were an aggregate of m countries with an underlying Ricardian technology. And a system of countries i = 1,..., k acts as if it were a system of $m_1 + \ldots + m_k$ Ricardian countries. Thus the Ricardian assumptions underlying this paper – embodied in equations (1), (2), (3) – may have more scope than meets the eye.

In any case, a more complete model involves adding a demand sector and some dynamics. This opens a mine of possibilities. In dynamic models of technological change, will some technology equivalence classes be favored over others? Is there a tendency to move from one to another?

Another approach is to take one technology type and emulate Dornbusch, Fischer and Samuelson (1977) in going to a continuum of industries or countries or both. For two countries there is essentially just one generic technology class, and the way to let $n \to \infty$ is clear; similarly for two industries and $m \to \infty$. For m, $n \geq 3$ we must not only choose a technology class, but make sense of the notion of m or n or both going to the limit.

Fortunately there is one case where this can be done in a natural way, and that is for the simplest possible class of technologies in general position. For $n = 3$ the class is given by (34) (more precisely, by (34) with the rows multiplied by arbitrary constants) and yields the McKenzie tiling where all quadrilaterals are U-faces.

For the general definition, with countries $i = 1,\ldots, m$, industries $j = 1,\ldots, n$ (the ordering by labels is essential), technology A has simple structure if, for all countries h, i and industries j, k, with $h < i$ and $j < k$, we have

$$a_{hj}a_{ik} < a_{ij}a_{hk} \tag{37}$$

We give some characterizations. For distinct countries $i_1,\ldots, i_r$, and industries $j_1,\ldots,j_r$, where $i_1 <\ldots< i_r$, the match $\{(i_1,j_1),\ldots, (i_r,j_r)\}$ is downward sloping if $j_1 <\ldots< j_r$.

Theorem 36. For m-by-n technology A, each of the following conditions implies the others:

(i) A has simple structure.

(ii) Every Jones match is downward sloping.

(iii) Every match in every viable pattern S is downward sloping.

Proof. (i) <u>implies</u> (ii): Suppose (ii) false, so some Jones match, say {(1,j), (2,k),..., (r,q)} is not downward sloping — say j > k. The submatch {(1,j), (2,k)} is still Jones, so $a_{1j}a_{2k} \le a_{1k}a_{2j}$, contradicting (37).

(ii) <u>implies</u> (iii): By the generalized McKenzie-Jones principle every match in viable S is Jones.

(iii) <u>implies</u> (i): Suppose (i) false, so that, say, $a_{11}a_{22} \ge a_{21}a_{12}$. Define S by $s_{12} = 1 = s_{21}$, and zero elsewhere. Choose $(P_1, \ldots, P_n)$ by: $a_{11}/a_{12} \ge P_1/P_2 \ge a_{21}/a_{22}$, with $P_3, \ldots, P_n$ very small. This price system supports S, so S is viable, contradicting (iii). ▒

That A is in general position follows from property (ii) since a square submatrix has only one downward sloping match (its "main diagonal"). It is also easy to see that every downward sloping match is a Jones match (since any other match can have its score lowered by interchange, by (37)), and from this, that any pattern S having all its matches downward sloping is viable. (Incidentally, it suffices that (37) holds for i = h + 1, k = j + 1 — see Yule and Kendall, p. 57).

For m and n given, the technologies of simple structure constitute a single equivalence class of technologies, since they — and only they — have the same Jones matches. It is easy to find examples of these technologies, the simplest being $a_{ij} = i + j$.

The extension to a continuum of countries or industries or both is now clear. Suppose for example that both are indexed by the real numbers. Technology A is then a function of two real variables. It has <u>simple structure</u> if, for all countries u, v and all industries x, y, where u < v and x < y, we have

$$a(u,x)a(v,y) < a(v,x)a(u,y) \tag{38}$$

an obvious generalization of (37).

Must we then work out an elaborate theory extending the results of this paper to the continuous case? On the supply side the answer is no, because the theory already exists! It is the theory of Thünen systems in location theory (Faden, Chapter 8).

This requires some explanation. The key observation is Theorem 2 above, which characterizes viable patterns in terms of the linear programming transportation problem. What happens when the theory of comparative advantage extends to a continuum of countries and/or industries? Well, there is a natural generalization from the ordinary to the <u>measure-theoretic</u> transportation problem (Faden, Chapter 7), and these ideas also carry over in a natural way to the theory of comparative advantage. (The real line is best thought of here as a measure space.)

We have no time to give the full story here, but a few examples will illustrate how this generalization proceeds. Labor resources ($L_1, \ldots, L_m$) go over to a resource measure on country space C. Outputs ($X_1, \ldots, X_n$) go over to an output measure on industry space I. Allocations (L_{ij}), (X_{ij}) are measures on the product space C × I. Equations (1) and (2) are expressed in terms of marginals. Equation (3) states that the technology function A is the density connecting these measures. The pattern S of an allocation is now its <u>support</u> – the subset of C × I that tells, roughly, where the allocation measure sits.

Furthermore, Theorem 2 and others generalize too. The measure-theoretic transportation problem has a dual, and its solution (after antilogging) furnishes P′s and W′s satisfying (4) and (5). The concept of Jones match generalizes. In short, the entire theory carries over in its basic structure.

To continue the story, define $t(u,x) = \log a(u,-x)$. Then (38) becomes, for $u < v$, $x < y$.

$$t(u,x) + t(v,y) > t(v,x) + t(u,y) \qquad (39)$$

which is called "positive cross-differences" (Faden, p. 421). The theory of Thünen systems arises when (39) is imposed on the cost function in the measure-theoretic transportation problem. This special case corresponds exactly to the "continuous" theory of comparative advantage under a technology of simple structure: "Downward-sloping" goes over to "weight-falloff", etc. Summarizing all this in an "analogy" diagram:

Comparative Advantage	Transportation Problem
finite CA	ordinary TP
continuum CA	measure-theoretic TP
continuum with simple structure	Thünen theory

To carry out the "continuum" strategy of Dornbusch, et. al., for the many-country, many-industry case requires only that we add a demand side to the supply side that comes ready-made from the theory of Th nen systems.

Ironically, if we actually add transportation costs (or tariffs) to the standard comparative advantage model the correspondence breaks down: The "law of one price" fails; P_j must be written as P_{ij} and the entire theory in its present form collapses. Nonetheless, the structural connection between the theory of comparative advantage and location theory has been established.

References

Berge, Claude, Graphs and Hypergraphs (North-Holland, Amsterdam, rev. ed., 1976).

Dornbusch, Rudiger, Stanley Fischer, Paul A. Samuelson, "Comparative Advantage, Trade and Payments in a Ricardian Model With a Continuum of Goods", American Economic Review 67: 823-839, Dec. 1977 (reprinted as Chapter 14 of R. Dornbusch, Exchange Rates and Inflation (MIT Pr., Cambridge, 1988)).

Faden, Arnold M., Economics of Space and Time: The Measure-Theoretic Foundations of Social Science (Iowa State University Press, Ames, 1977).

Feller, William, An Introduction to Probability Theory and Its Applications, Vol. 1 (Wiley, New York, 2nd ed., 1957).

Gale, David, The Theory of Linear Economic Models (McGraw-Hill, New York, 1960).

Graham, Frank D., The Theory of International Values (Princeton University Press, Princeton, 1948).

Haberler, Gottfried, The Theory of International Trade (Macmillan, New York, 1936).

Jones, Ronald W., "Comparative Advantage and the Theory of Tariffs: A Multi-Country, Multi-Commodity Model", Review of Economic Studies, 28: 161-175, June 1961 (reprinted as Chapter 3 of R. W. Jones, International Trade: Essays in Theory (North-Holland, Amsterdam, 1979)).

Kuhn, Harold W., "Lectures on Mathematical Economics", pp. 49-84 of Mathematics of the Decision Sciences, Part 2, G. B. Dantzig & A. F. Veinott, Jr. (editors) (Amer. Math. Soc., Providence, 1968).

McKenzie, Lionel W., "Specialization and Efficiency in World Production", Review of Economic Studies 21: 165-180, June, 1954.

McKenzie, Lionel W., "International Trade, II. Mathematical Theory", International Encyclopedia of the Social Sciences, (Macmillan, 1968), vol. 8.

Murty, Katta G., Linear Programming (Wiley, New York, 1983).

Ruffin, Roy J., "The Missing Link: The Ricardian Approach to the Factor Endowments Theory of Trade", American Economic Review 78: 759-772, Sept. 1988.

Viner, Jacob, Studies in the Theory of International Trade, (Harper, New York, 1937).

Yule, G. Udny & M. G. Kendall, An Introduction to the Theory of Statistics, (Griffin, London, 14th ed., 1950).