Simplicial Geometry: a Barycentric Approach

Arnold Faden

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The elementary geometry of triangles includes properties such as the existence and uniqueness of medians, in-centers, orthocenters, 9-point centers, formulae for these, relations amongst these, etc. Now, triangles are 2-simplexes, and the question arises, which of these properties carry over to arbitrary n-simplexes for any dimensions and in what form. For example, are there analogues of the 9-point circle for any n-simplex and if not, what properties are necessary and sufficient for such analogues to exist?

Simplexes are thought of here as embedded in Euclidean space, but the emphasis will be on their intrinsic properties as determined by the metric among their vertices. The barycentric (or affine) approach fits naturally here.

Formally, an $n-1$ simplex is given by n points $x_1, \dots x_n$, in $\mathbb{R}^{n-1}$ that are in general position. This means that the n points $\binom{x_1}{1}, \dots \binom{x_n}{1}$ in $\mathbb{R}^n$ are linearly independent.

(Here $\binom{x_i}{1}$ appends an extra 1 component to the column vectors $x_i \dots$. An equivalent definition is that the $n-1$ points $(x_2 - x_1), (x_3 - x1), \dots (x_n - x_1)$ are linearly independent in $\mathbb{R}^{n-1}$.)

Given this set-up, any point $y \in \mathbb{R}^{n-1}$ is uniquely expressible as an affine combination $y = x_1\alpha_1 + \dots x_n\alpha_n$, where $\alpha_1 + \dots + \alpha_n = 1$. (Proof. $\binom{x_1}{1}, \dots \binom{x_n}{1}$ are a basis in $\mathbb{R}^n$, hence $\binom{y}{1}$ is uniquely expressible as $\binom{x_1}{1}\alpha_1 + \dots + \binom{x_n}{1}\alpha_n$. The last line reads $1 = \alpha_1 + \dots + \alpha_n$. QED.)

(Note that we are using the terms affine and barycentric synonymously though the latter usually carries the additional requirement $\alpha_i \geq 0, i = 1, \dots n$ (the "inside" of the simplex). No such assumption is made here since, among other things, several of the centers mentioned above can lie outside the simplex. In any case, the $\alpha_1, \dots \alpha_n$ will be called the affine or barycentric coordinates of y.)

Let B be the space of all affine n-tuples $\{\alpha \mid \alpha_1 + \dots + \alpha_n = 1\}$. Also, let Z be the space $\{\alpha \mid \alpha_1 + \dots + \alpha_n = 0\}$. ($Z$ for zero-sum.) Note that Z is a subspace of $\mathbb{R}^n$, the hyperplane of all n-tuples orthogonal to $\omega = (1, \dots 1)$ and B is a hyperplane parallel to Z.

Consider the map $L : Z \to \mathbb{R}^{n-1}$ given by $\alpha \mapsto x_1\alpha_1 + \dots + x_n\alpha_n$. This is a linear isomorphism. (Proof: Linearity is clear. Isomorphism follows from a proof similar to the one above – Just substitute $\binom{y}{0}$ for $\binom{y}{1}$. QED.)

Now $\mathbb{R}^{n-1}$ has the usual inner product and metric, which can be pulled back to Z via the isomorphism given above. This turns out to be a key insight.

Theorem 1. *Let A be the $n \times n$ matrix given by $a_{ij} = d_{ij}^2$, the squared Euclidean distance from vertex x_i to x_j. Then there exist $\alpha, \beta \in Z$, such that*

$$\langle \alpha, \beta \rangle = -\frac{1}{2}\alpha' A \beta \tag{1}$$

Proof.

$$a_{ij} = (x_i - x_j)'(x_i - x_j) = |x_i|^2 + |x_j|^2 - 2x_i x_j$$

Hence,

$$-\frac{1}{2}\alpha' A\beta = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i(|x_i|^2 + |x_j|^2)\beta_j + (\sum x_i\alpha_i)'(\sum x_j\beta_j)$$
$$= (L\alpha)'(L\beta) = \langle \alpha, \beta \rangle$$

The double symmetries drop out by $\alpha, \beta \in Z$. QED. □

Equation (1) gives the translation between barycentric coordinates and matrices. For let $\alpha, \beta \in B$ be the coordinates of points $y_\alpha, y_\beta \in \mathbb{R}^{n-1}$. Then $\gamma \in Z$ where $\gamma = \beta - \alpha$. The squared distance from y_α to y_β is then $|x_1\gamma_1 + \ldots + x_n\gamma_n|^2 = \langle \gamma, \gamma \rangle = -\frac{1}{2}\gamma' A \gamma$.

Concrete example: Take a triangle with sides a, b, c. What is the distance from the vertex opposite side c to the midpoint of side c? The coordinates of these points are $(0, 0, 1)$ and $(\frac{1}{2}, \frac{1}{2}, 0)$. Hence the squared distance is

$$-\frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}, -1\right)\begin{pmatrix} 0 & c^2 & a^2 \\ c^2 & 0 & b^2 \\ a^2 & b^2 & 0 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix} = \frac{1}{2}a^2 \cdot \frac{1}{2}b^2 - \frac{1}{4}c^2$$

As an exercise, show that the squared distance between x_i and x_j is indeed a_{ij}.

Now consider the point with coordinates $\beta \in B$, and compute the mean square distance of y_β from the n vertices, where the mean is taken with respect to the system of weights $\alpha \in B$.

Theorem 2. *The mean squared distance $msd = -\frac{1}{2}\beta' A\beta + \alpha A\beta$ $(*)$. If α is fixed, this attains a unique minimum at $\beta = \alpha$, with value $\frac{1}{2}\alpha' A\alpha$.*

Proof. Vertex x_i has coordinates δ_i, which equals 1 in place i, 0 elsewhere, hence the squared distance is $-\frac{1}{2}(\beta - \delta_i)' A(\beta - \delta_i) = -\frac{1}{2}A\beta + a_i'\beta$, where a_i is the i^{th} row of A. Multiply by α_i and add over i to get $(*)$.

Note that $(*)$ may be rewritten as

$$-\frac{1}{2}(\beta - \alpha)' A(\beta - \alpha) + \frac{1}{2}\alpha' A\alpha \tag{2}$$

which is minimized at $\beta = \alpha$ since the first term is the inner product $\langle \beta - \alpha, \beta - \alpha \rangle$. QED. □

Expression (2) is rather interesting because it decomposes the msd $(*)$ into the sum of its lowest possible value $\frac{1}{2}\alpha' A\alpha$ and the squared distance from the minimum point y_α to y_β. This is reminiscent of the theorem that the msd of a probability distribution is minimized at the mean. In fact, it generalizes that result in a way in that negative components of α are allowed: actually, $\frac{1}{2}\alpha' A\alpha$ takes on negative values sometimes.

We will refer to $\frac{1}{2}\alpha' A\alpha$ as the variance of distribution α, $\alpha \in B$.

Suppose two agents play a zero-sum game on an $n-1$ simplex. The Max player goes first, choosing $\alpha \in B$. Then the Min player chooses $\beta \in B$, the payoff being $(*)$. Obviously, the Min Player chooses $\beta = \alpha$. But what does the Max player choose?

Theorem 3. *Let C be the circumcenter of a simplex and let R be the circumradius. Then, for any $\beta \in B$,*

$$\frac{1}{2}\beta' A\beta = R^2 - \Big(\textit{distance}^2 \textit{ from } y_\beta \textit{ to } C\Big) \tag{3}$$

Proof. Let $\alpha \in B$ be the coordinates of C. Then (reversing the roles of α, β in (2)), the msd of C with weights $\beta \in B$ is $-\frac{1}{2}(\alpha-\beta)' A(\alpha-\beta) + \frac{1}{2}\beta' A\beta$. But this msd must equal R^2 for any $\beta \in B$, since the distances from C to each vertex are the same. This yields (3). QED. □

Evidently, the Max player will choose α_i, getting a payoff of R^2. But (3) says a lot more. For any point on the hyper-circumsphere with coordinates $\beta \in B$, the variance $\frac{1}{2}\beta' A\beta = 0$. In particular, this holds for the vertices themselves. Consider a triangle with sides a, b, c. A theorem of Euler states that the distance^2 from circum- to incenter is $R^2 - 2Rr$, where r is the inradius. But this follows from (3). Note that the coordinates of the incenter are $(a, b, c)/(a + b + c)$ and the variance works out to be $abc/(a + b + c) = 2Rr$. Similarly, the distance^2 from circumcenter to median is $R^2 - 1/a(a^2 + b^2 + c^2)$.

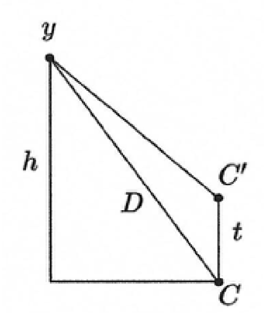

Fig 1

We now remove the condition in the preceding theorem:

Theorem 4. *For any simplex, the circumcenter exists and is unique.*

Proof. Clearly this holds for $n = 2$ points. Now proceed by induction on n. Assume for any simplex of n points in $\mathbb{R}^{n-1}$ that there is a unique circumcenter $C \in \mathbb{R}^n$ (the horizontal line in Fig 1). Now go to $\mathbb{R}^n$ and add a new point y at distance h from $\mathbb{R}^{n-1}$ (with $h \neq 0$ to preserve affine independence). Consider the line through C perpendicular to $\mathbb{R}^{n-1}$. Any point on this line remains equidistant from the n old points by Pythagorus. (The new common spherical distance $= R^2 + t^2$, where R is the old circumradius.) Further, any point off this perpendicular cannot be equidistant from the n old points, for if it were then its projection on $\mathbb{R}^{n-1}$ would also be equidistant contradicting the uniqueness of C. It remains to find the point C' at distance t from C that equates these distances with the distance to y:

$$R^2 + t^2 = (t-h)^2 + D^2 - h^2, \text{ yielding } t = (D^2 - R^2)/2h$$

a unique solution. QED. □

Below we will give another proof of this theorem that yields explicit formulae for C and R.

Lemma 1. *Matrix A is invertible.*

Proof. Suppose that $Ax = 0$ where x is an $n \times 1$ vector. First assume that $u'x \neq 0$ where $u' = (1, 1, \ldots 1)$. By scaling, we may assume $u'x = 1$, that is, $x \in B$. Now let $w \in B$, distinct from x, and with all components positive. Then $\frac{1}{2}w'Aw$ and $-\frac{1}{2}(w-x)'A(w-x)$ are both positive. Yet $Ax = 0$ implies that they add to zero. Contradiction. Thus, we must have $u'x = 0$, that is, $x \in Z$. But then $\langle x, x\rangle = -\frac{1}{2}x'Ax = 0$ so $x = 0$. QED. □

Theorem 5. *Let $\alpha \in B$ be the coordinates of circumcenter C and R the circumradius. Then $\alpha = A^{-1}u/u'A^{-1}u$ and $2R^2 = 1/u'A^{-1}u$, where again $u = (1, 1, \ldots 1)$.*