THE ABSTRACT TRANSPORTATION PROBLEM

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1. Introduction

In this paper we generalize the transportation problem of linear programming to the case of a possibly infinite number of sources and sinks.

Why bother to do this? In the first place, the surface of the earth is a continuum, and we may always think of transportation as a re-distribution of mass from one portion of this surface to another. Many transportation problems appearing in the literature achieve their finite character by a lumping together of continuum into a finite number of pieces, which are treated as points: for example, treating countries as single points in international trade models.

However, the possible realm of application of the abstract transportation model goes well beyond transportation per se. It is well known that a wide variety of models -- of resource allocation, scheduling, etc. -- can be thrown into the transportation format. Problems of the "caterer" form, for example, involve the redistribution of mass from one point in time to another. Since time is a continuum, such problems are often most naturally formulated with a continuum of origins and distinations. Again, there is an infinity of types of possible commodities or industrial processes; thus, a problem of the form, "how shall I assign my resources among various activities?", is again a problem with an infinity of sources and sinks.

Although this paper is not concerned directly with numerical applications, one possible "practical-man's" objection should be laid to rest. It is <u>not</u> true that the abstract transportation model must be approximated finitely -- by the lumping process mentioned above -- to achieve numerical results. What practice requires is that the set of possible answers be represented by a parameter space of relatively low dimension. This can be achieved by the lumping process, but can also be achieved in other ways, depending on the particular problem. This is illustrated by statistics, where continuous distributions are put to practical use by working with families of them indexed by a small number of parameters.

2. Formulating the model

The transportation problem with m sources and n sinks is --

Find mn non-negative numbers x_{ij} $(i = 1, \ldots m,\ j = 1, \ldots n)$ satisfying

$$\Sigma_j \; x_{ij} \leq a_i \qquad (i = 1, \ldots, m), \tag{1}$$

$$\Sigma_i \; x_{ij} \geq b_j \qquad (j = 1, \ldots, n), \tag{2}$$

and minimizing

$$\Sigma_{i,j} \; r_{ij} \; x_{ij}. \tag{3}$$

Here a_i is the capacity of source i, b_j is the requirement at sink j, and r_{ij} is the cost incurred per unit shipped from source i to sink j.

The natural generalization of this uses measures and integrals. We now define the necessary concepts.[1]

$x \in E$ signifies that x is a member of E. Set E is contained in Set F, written $E \subset F$, if every member of E is a member of F. The union of sets E and F, written $E \cup F$, is the set whose members are members of E or F or both. More generally, if $\mathcal{S}$ is an arbitrary collection of sets, its union, written $\cup\mathcal{S}$, is the set whose members are members of at least one of the sets of $\mathcal{S}$. The intersection of sets E and F, written $E \cap F$, is the set whose members are members of both E and F. If $\mathcal{S}$ is an arbitrary collection of sets, its intersection, written $\cap\mathcal{S}$, is the set whose members are members of all of the $\mathcal{S}$ sets. The complement of set F in set E, written $E \setminus F$, is the set whose members are members of E but not of F. The empty set, written $\emptyset$, is the set which has no members.

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[1]This paper is self-contained so far as definitions go, but standard theorems are quoted without proof.

A set E is countable if its members can be enumerated in a finite or infinite sequence $\{e_1, e_2, \ldots\}$.

Let A be a fixed set, and Σ a collection of subsets of A; Σ is a sigma-field over A if (1) $A \varepsilon \Sigma$, and (2) if $E \varepsilon \Sigma$, then $(A \backslash E) \varepsilon \Sigma$, and (3) if $\mathcal{G}$ is any countable collection of Σ-sets, then $\cup \mathcal{G} \varepsilon \Sigma$. The pair (A, Σ) is called a measurable space, and the members of Σ are called measurable sets.

Let (A, Σ') and (B, Σ'') be two measurable spaces. The product measurable space, written $(A \times B, \Sigma' \times \Sigma'')$ is defined as follows. A x B is the cartesian product of A and B, the set of all ordered pairs (x, y), $(x \varepsilon A, y \varepsilon B)$. Any subset of A x B of the form E x F, $(E \varepsilon \Sigma', F \varepsilon \Sigma'')$ is called a measurable rectangle. $\Sigma' \times \Sigma''$ is the class of all sets common to all sigma-fields over A x B which have all measurable rectangles as members. $\Sigma' \times \Sigma''$ itself can be shown to be a sigma-field over A x B, the sigma-field generated by the measurable rectangles.

A bounded, non-negative function, μ, whose domain is a sigma-field, is a measure if $\mu(\cup \mathcal{G}) = \sum_{n=1}^{\infty} \mu(G_n)$ whenever $\mathcal{G}$ is a countable collection $\{G_1, G_2, G_3, \ldots\}$ of measurable sets which are disjoint ($G_m \cap G_n = \emptyset$ for all $m \neq n$).[2] A probability on (A, Σ) is a measure μ with domain Σ for which $\mu(A) = 1$.

The notation $\{x | \ldots\}$ represents the set of all x having the property stated after the bar. For example, $\{x | f(x) > c\}$ is the set of all x for which the value of a certain function f exceeds the number c. Given a measurable space (A, Σ), a function f with domain A is said to be measurable, if the sets $\{x | f(x) > c\}$ are measurable for all real numbers c. It can be shown that the substitution of "$\geq$", "$\leq$", or "$<$" for "$>$" in this definition yields the same set of functions.

[2] The Σ in this sentence stands for summation, not for a class of sets. The distinction will be clear from the context.

If f is a bounded measurable non-negative function, and μ a measure, both with respect to (A, Σ), the integral of f with respect to μ, written $\int_A f\, d\mu$, is defined as $\int_0^\infty \mu\{x \mid f(x) > t\}\, dt$, where the integral on the right is an ordinary Riemann integral of the indicated (monotone decreasing) function on the real line. If f takes on negative values, we split it into its positive and negative parts: $f(x) = \max(f(x), 0) - \max(-f(x), 0)$, take the integral of each part, and subtract.[3]

We are now ready to formulate the abstract transportation problem: Given two triples, (A, Σ', μ') and (B, Σ'', μ''), and a real-valued function r with domain $A \times B$, where

(1) Σ' is a sigma-field over set A, and μ' is a measure on Σ'; and similarly for Σ'', B, and μ'';

(2) r is bounded, and measurable, with respect to the product sigma-field $\Sigma' \times \Sigma''$ over $A \times B$;

find a measure $\emptyset$ on $(A \times B, \Sigma' \times \Sigma'')$ which satisfies

$$\emptyset(E \times B) \leq \mu'(E) \quad \text{for all } E \in \Sigma', \tag{4}$$

$$\emptyset(A \times F) \geq \mu''(F) \quad \text{for all } F \in \Sigma'', \tag{5}$$

and which minimizes $\int_{A \times B} r\, d\emptyset$ over all such measures. (6)

This bears direct comparison with the finite transportation problem (1), (2), (3). A and B are the origin and destination spaces, respectively. μ' is the capacity measure. The constraint (4), which is a direct generalization of (1), states that $\emptyset(E \times B)$, which is the total flow out of region E, cannot exceed $\mu'(E)$, the capacity of the sources in region E. μ'' is the requirement measure, and (5), the generalization of (2), states that $\emptyset(A \times F)$, the total inflow into region F, must at least meet the requirement for that

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[3]For further reading in measure theory, the reader is referred to P.R. Halmos, Measure Theory (Princeton: Van Nostrand, 1950).

region, $\mu''(F)$. ϕ is the unknown flow from origin to destination: $\phi(E \times F)$ equals the total mass flowing from region E to region F. r generalizes the unit cost function.

A careful check of the definitions shows that, in the special case where A and B are finite sets, and Σ', Σ'' are the classes of all subsets of A and B, respectively, (4), (5), (6) reduce to (1), (2), (3).

Before going on to the analysis of the abstract transportation problem, let us look at some related work. Martin Beckmann has worked on some related but non-overlapping problems.[4] Beckmann makes essential use of the topology of 2-dimensional Euclidean space, using vector methods (gradients, curls, etc.). In this sense his is a special case of ours. On the other hand he deals with the entire flow field, whereas we restrict our attention just to origin-destination connections. Thus he is dealing essentially with a <u>transshipment</u>, rather than a transportation problem.

The abstract transshipment problem differs from the abstract transportation problem as follows: the spaces A and B, and the sigma-fields Σ' and Σ'', are identical; let us write them as (A, Σ). There is a <u>net requirement</u> (signed) measure, μ, on (A, Σ). This differs from an ordinary measure only in that it may take on negative values. The constraints (4) and (5) are replaced by:

$$\phi(A \times E) - \phi(E \times A) \geq \mu(E) \quad \text{for all } E \in \Sigma. \qquad (7)$$

The transshipment problem, then, is to find a measure ϕ on $(A \times A, \Sigma \times \Sigma)$ which satisfies (7), and minimizes (6) over all such measures.

In the finite case there is a well known procedure for reducing transshipment to transportation problems.[5] This procedure breaks down when the number

[4] "A continuous model of transportation", <u>Econometrica</u>, 20: 643-660, October, 1952; "the partial equilibrium of a continuous space market", <u>Weltwirtschaftliches Archiv</u>, 71: 73-87, 1953.

[5] A. Orden, "The transshipment problem", <u>Management Science</u>, 2: 276-285, April, 1956.

of sets in Σ is infinite. The transshipment problem is essentially distinct from (and more difficult than) the transportation problem in the general case. We shall concentrate our attention on the latter; it should be pointed out, however, that several of the theorems we derive have analogs for the transshipment problem.

The true locus classicus for the abstract transportation problem is found in the work of L.V. Kantorovich.[6] He deals with the problem of (4) - (6) except for two minor points: the constraints are taken to be equalities, and A is identified with B. (This last identification involves no real loss of generality, but can be misleading, as we shall see.) He discusses the existence of optimal solutions and their connection with "potentials" (that is, dual prices, in the terminology which developed later on) -- a remarkable achievement for its time. Kantorovich's article is peculiar in several respects. It is all of three pages long, and written with extreme brevity and apparent haste. In fact, the major theorem -- stating the existence of dual prices associated with an optimal flow -- is false, as one can show by a simple counter example.[7] (The root of the error, by the way, lies in the fact that the constraints imposed are those of the transportation problem, while the dual prices are defined in a way appropriate to the transshipment problem. If the unit cost function r violates the triangle inequality -- as it well might -- one can get a counter-example, as footnote 7 illustrates). More surprising is the fact that the method of proof used appears to be insufficient to prove the corrected version of the theorem.

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[6] "On the translocation of masses", Management Science, 5: 1-4, October, 1958 (originally published in Doklady Nauk USSR, 37, #7-8, 199-201, 1942).

[7] Space contains 3 points $\{x, y, z\}$; the capacity at point x equals one; the requirement at point z equals one; all other capacities and requirements equal zero; $r(x, y) = 1$, $r(y, z) = 1$, $r(x, z) = 3$; all other r's arbitrary; the only feasible, hence optimal, flow is one unit from x to z; there is no Kantorovich potential for this flow, since it must satisfy the incompatible relations: $U_z - U_x = 3$; $U_z - U_y \leq 1$; $U_y - U_x \leq 1$.

Our aim in the bulk of this paper is to go over the ground sketched out by Kantorovich, to derive in a rigorous fashion conditions for the existence of optimal solutions, and to give a "pseudo-constructive" method for finding dual prices from an optimal solution.

3. Feasibility

We begin the investigation of the abstract transportation problem, (4) - (6), with a simple feasibility result.

Theorem 1: There exists a feasible solution to the constraints (4), (5) iff

$$\mu'(A) \geq \mu''(B). \tag{8}$$

Proof: If ϕ is a feasible solution, then $\mu'(A) \geq \phi(A \times B) \geq \mu''(B)$, so the stated condition is necessary.

Conversely, let $\mu'(A) \geq \mu''(B)$; if $\mu'(A) = 0$, both μ' and μ'' are identically zero, and then the identically zero measure on $\Sigma' \times \Sigma''$ is obviously feasible. If $\mu'(A) > 0$, define the function ϕ on measurable rectangles E x F by: $\phi(E \times F) = \dfrac{\mu'(E) \cdot \mu''(F)}{\mu'(A)}$. (9)

It is a standard measure theorem that such a "product" function can be extended to a measure over the product space $(A \times B, \Sigma' \times \Sigma'')$. One checks immediately that this measure is feasible, since

$$\phi(E \times B) = \frac{\mu'(E) \cdot \mu''(B)}{\mu'(A)} \leq \mu'(E), \text{ and}$$

$$\phi(A \times F) = \mu''(F). \qquad \text{QED}$$

This may be stated: a feasible solution exists iff total capacity of sources at least matches total requirements of sinks. This well known result for the finite case thus carries over in general.

We are also interested in the abstract transportation problem for the case where the constraints in (4) and (5) are stated as equalities:

$$\phi(E \times B) = \mu'(E) \quad \text{for all } E \in \Sigma' \tag{10}$$

$$\phi(A \times F) = \mu''(F) \quad \text{for all } F \in \Sigma'' \tag{11}$$

Theorem 2: If $\mu'(A) = \mu''(B)$, then any feasible solution to (4), (5) satisfies these constraints with equality (that is, it is in fact feasible for the stricter constraints (10), (11)).

Proof: Suppose, for example, that some constraint in (4) is satisfied with strict inequality: $\emptyset(G \times B) < \mu'(G)$ for some $G \varepsilon \Sigma'$; then $\mu''(B) \leq \emptyset(A \times B)$ $= \emptyset(G \times B) + \emptyset((A \setminus G) \times B) < \mu'(G) + \mu'(A \setminus G) = \mu'(A)$, a contradiction. The proof for a strict (5) inequality is similar. QED

We now have an equally simple feasibility result for the equality-constrained case:

Theorem 3: There exists a feasible solution to the constraints (10), (11), iff

$$\mu'(A) = \mu''(B). \tag{12}$$

Proof: If $\emptyset$ is feasible, then $\mu'(A) = \emptyset(A \times B) = \mu''(B)$. If $\mu'(A) = \mu''(B)$, then Theorem 1 tells us that there is a feasible solution to (4) and (5), and Theorem 2 that these constraints are satisfied as equalities. QED

4. Duality

Every finite linear program has a dual, and the dual of the transportation problem (1) - (3) is

Find non-negative numbers p_i $(i = 1, \ldots, m)$ and q_j $(j = 1, \ldots, n)$ satisfying

$$q_j - p_i \leq r_{ij} \quad (i = 1, \ldots, m; \; j = 1, \ldots, n) \tag{13}$$

and maximizing: $$\Sigma_j \, q_j \, b_j - \Sigma_i \, p_i a_i \, . \tag{14}$$

The dual of the transportation problem with equality constraints is the same as this, except that p_i and q_j are not constrained to be non-negative.

Analogously, we define the dual of the abstract transportation problem (4) - (6) to be -

Find a bounded, non-negative, function, p, with domain A, measurable with respect to Σ', and a bounded, non-negative, function, q, with domain B, measurable with respect to Σ'', satisfying

$$q(y) - p(x) \leq r(x, y) \quad \text{for all } x \in A,\ y \in B. \tag{15}$$

and maximizing:

$$\int_B q \, d\mu'' - \int_A p \, d\mu'. \tag{16}$$

The dual of the abstract transportation problem with equality constraints is defined to be the same as this, except that p and q need not be non-negative.

Apart from the obvious formal similarities between these abstract duals and the finite duals, many of the standard relations between primal and dual carry over to the general case. We first define one more concept.

Given a measure ϕ on $(A \times B, \Sigma' \times \Sigma'')$, its left-marginal measure, ϕ', is the measure defined on (A, Σ') by the rule:

$$\phi'(E) = \phi(E \times B), \quad \text{all } E \in \Sigma'. \tag{17}$$

Similarly, ϕ'', the right-marginal measure of ϕ, is defined on (B, Σ'') by the rule:

$$\phi''(F) = \phi(A \times F), \quad \text{all } F \in \Sigma''. \tag{18}$$

(If ϕ is a probability, its marginals coincide with the usual notion of marginal probabilities.) In terms of marginals, the basic transportation constraints (4) and (5) assume the simple form -

$$\phi' \leq \mu', \quad \text{and} \tag{19}$$

$$\phi'' \geq \mu''. \tag{20}$$

Theorem 4: If ϕ is feasible for the abstract transportation problem, (4) - (5), and (p, q) is feasible for the dual, (15), then

$$\int_{A \times B} r \, d\phi \geq \int_B q \, d\mu'' - \int_A p \, d\mu'. \tag{21}$$

Proof: We adopt the simple convention that "p" stands both for a function with domain A, and for the function with domain A x B defined by $p(x, y) = p(x)$, all $x \in A$, $y \in B$; similarly, "q" stands for two functions, with domains B, and A x B, related by $q(x, y) = q(y)$; which function we are talking about is clear from the domain of integration. Then

$$\int_{A\times B} r\, d\phi \geq \int_{A\times B} (q-p)d\phi = \int_{A\times B} q\, d\phi - \int_{A\times B} p\, d\phi =$$

$$\int_B q\, d\phi'' - \int_A p\, d\phi' \geq \int_B q\, d\mu'' - \int_A p\, d\mu'. \tag{22}$$

(The first inequality follows from (15), the equalities reflect standard integration theorems, and the last inequality follows from (19) and (20), together with the fact that p and q are non-negative.) QED

Theorem 5: If ϕ is feasible for the abstract transportation problem with equality constraints, and (p, q) is feasible for the dual of this problem, then (21) is still valid.

Proof: Same as above, except that the last "$\geq$" should be replaced by "=". QED

These theorems carry over the fact that, in a pair of linear programs, the value of the maximum program never exceeds the value of the minimum program, for any pair of feasible values.

We are now interested in conditions under which the inequality (21) becomes an equality, because, in view of theorems 4 and 5, this would guarantee that ϕ, and (p, q) are optimal for their respective programs.

Definition: Let ϕ be feasible for the abstract transportation problem (4) - (5), and $(p, q) \geq 0$ feasible for the dual problem (15). (p, q) is a measure potential for ϕ if the following three conditions are satisfied:

$$\phi \{(x, y) | q(y) - p(x) < r(x, y)\} = 0, \tag{23}$$

$$\phi' \{x | p(x) > 0\} = \mu' \{x | p(x) > 0\}, \text{ and} \tag{24}$$

$$\phi'' \{y | q(y) > 0\} = \mu'' \{y | q(y) > 0\}. \tag{25}$$

(23) states that there is no flow on the set of source-sink pairs for which (15) is satisfied with strict inequality. (24) states that capacity is used completely on the set of sources for which $p > 0$. (25) states that requirements are just met on the set of sinks for which $q > 0$.

The same definition also serves for the pair of equality-constrained programs, except that (p, q) need not be non-negative. Note also that for equality constraints, (24) and (25) are automatically fulfilled, so that they may be omitted from the definition.

Theorem 6: For a given feasible pair, $\emptyset$ and (p, q), relation (21) is an equality iff (p, q) is a measure potential for $\emptyset$. (This applies both to the inequality and the equality-constrained programs.)

Proof: Examining the chain of relations (22), we find that the first "$\geq$" becomes an equality iff (23) is fulfilled, and the last "$\geq$" becomes an equality iff (24) and (25) are fulfilled. QED

Corollary: If $\emptyset$ and (p, q) are feasible for their respective programs, and (p, q) is a measure potential for $\emptyset$, then both are optimal for their programs.

The definition and theorem 6 generalizes the familiar "complementary slackness" conditions of linear programming, according to which equality is attained in dual program values iff strict inequality in one program's constraints is matched by a zero value of the corresponding variable in the other. The further (and deeper) result in finite linear programming theory that "complementary slackness" is a necessary condition for optimality, does not necessarily carry over to the infinite case.

5. Existence of optimal solutions

Up to this point, measure-theoretic concepts have sufficed to define our concepts and prove our theorems. From here on topological concepts will also be needed. Indeed, the author does not know of any method of proving the existence of optimal solutions to the abstract transportation problem using measure-theoretic concepts alone. Also, we know of no way to construct measure potentials directly from optimal solutions. Instead we give a construction for the related notion of "topological potential", and under certain additional conditions these turn out to be measure potentials as well. The basic definitions follow.

Given a fixed set A, let $\mathcal{T}$ be a collection of subsets of A. $\mathcal{T}$ is a topology over A if (1) $A \in \mathcal{T}$, and the empty set $\emptyset \in \mathcal{T}$; (2) if $G_1 \in \mathcal{T}$ and $G_2 \in \mathcal{T}$, then $G_1 \cap G_2 \in \mathcal{T}$; (3) if $\mathcal{G}$ is any collection of $\mathcal{T}$-sets, then $\cup\mathcal{G} \in \mathcal{T}$. The pair (A, $\mathcal{T}$) is a topological space. The members of $\mathcal{T}$ are called the open sets. A set is closed if its complement in A is open.

Let (A, $\mathcal{T}'$) and (B, $\mathcal{T}''$) be two topological spaces. The product topological space, written (A x B, $\mathcal{T}' \times \mathcal{T}''$), is defined as follows. A x B is the cartesian product of A and B. Any subset of A x B of the form $G' \times G''$ ($G' \in \mathcal{T}'$, $G'' \in \mathcal{T}''$) is called an open rectangle. $\mathcal{T}' \times \mathcal{T}''$ is the class of all sets which are unions of an arbitrary number of open rectangles, together with the empty set. It can be shown that $\mathcal{T}' \times \mathcal{T}''$ is, indeed, a topology over A x B.

A topological space (A, $\mathcal{T}$) is separable if there is a countable set $E \subset A$ such that every non-empty open set has a member in common with E. A real-valued function f with domain A is continuous for the topology $\mathcal{T}$ if every set of the form $\{x / a < f(x) < b\}$, where a and b are real numbers, is open.

A metric on a set, A, is a real-valued function, d, with domain A x A, having the properties (1) $d(x, x) = 0$; (2) $d(x, y) > 0$ if $x \neq y$; (3) $d(x,y) = d(y, x)$; and (4) $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in A$. The pair (A, d) is a metric space. A sequence, $x_1, x_2, \ldots$, in a metric space converges to x if, for any positive number ϵ, there is an integer N, such that $d(x_n, x) < \epsilon$ for all $n > N$. A metric space is complete if, for any sequence $x_1, x_2, \ldots$ having the property that $d(x_m, x_n) < \epsilon$ whenever m and n exceed some integer N, depending on the positive number ϵ, there is an x to which the sequence converges. (Roughly, when the points of a sequence get indefinitely close to each other, they get indefinitely close to some fixed point of the space.)

Any metric on A determines a topology on A as follows: a set $E \subset A$ is open if, for every point $x \in E$ there is a positive number ϵ such that all points within distance ϵ of x are members of E. A topology which is generated by some metric in this way is said to be <u>metrizable</u>. If it is generated by some <u>complete</u> metric it is said to be <u>topologically complete</u>. A set K in a metrizable topological space $(A, \mathcal{T})$ is <u>compact</u> if any sequence of members of K has a subsequence which converges to a member of K. If the subsequence is merely known to converge to a member of A, the set is <u>relatively compact</u>.

We will need certain concepts which involve both measure-theoretic and topological notions. The <u>Borel field</u> of a topological space is the smallest sigma-field of which every open set is a member. (More exactly, it is the sigma-field to which a set belongs iff it belongs to every sigma-field to which all open sets belong.) The members of the Borel field are called <u>Borel sets.</u> Let $(A, \mathcal{T})$ be a metrizable topological space, let Σ be its Borel field, and let μ be a measure on Σ. μ is said to be <u>tight</u> if, for every positive number ϵ, there is a compact set K such that $\mu(A \setminus K) < \epsilon$. Let $\mathcal{M}$ be a collection of measures on Σ. $\mathcal{M}$ is <u>uniformly tight</u> if (1) there is a number M such that $\mu(A) \leq M$ for all measures $\mu \in \mathcal{M}$, and (2) for every positive number ϵ there is a compact set K such that $\mu(A \setminus K) < \epsilon$ for all $\mu \in \mathcal{M}$. (Note that this requires more than that each measure of $\mathcal{M}$ be individually tight.) Let μ^* be a measure on Σ, and $\mu_1, \mu_2, \ldots$ a sequence of measures on Σ; the sequence $\mu_1, \mu_2, \ldots$ is said to <u>converge weakly</u> to μ^* if, for every real-valued function f with domain A which is bounded, and continuous with respect to $\mathcal{T}$, we have

$$\lim_{n \to \infty} \int_A f \, d\mu_n = \int_A f \, d\mu^* \qquad (26)$$

(in the ordinary sense of limit of a sequence of real numbers). The set of measures $\mathcal{M}$ is <u>weakly relatively compact</u> if every sequence of measures in $\mathcal{M}$ contains a subsequence which converges weakly to some measure (not necessarily a member of $\mathcal{M}$).

Finally, we need one or two concepts concerning real numbers. The supremum (abbreviated "sup") of a set of real numbers is the smallest number not exceeded by any of the numbers in the set; the infimum ("inf") is the largest number which is not greater than any number of the set. (If the set has a greatest number, it is the supremum; similarly, if it has a least number, that number is the infimum.) Given a sequence of real numbers, x_1, x_2, ..., let y_k be the supremum of the subsequence beginning with the k-th term: x_k, x_{k+1}, ...; the limit of the sequence y_1, y_2, ... formed in this way is called the lim sup of the original sequence. Lim inf of the original sequence is defined in the same way from the sequence of infima.

With these definitions taken care of, we are ready to proceed. The following two basic results from the theory of weak convergence play an essential role in our existence proofs.[8]

Lemma 1: (Prohorov-Varadarajan) Let $\mathcal{M}$ be a collection of measures on the Borel field of a metrizable topological space. If $\mathcal{M}$ is uniformly tight, then $\mathcal{M}$ is weakly relatively compact.

Lemma 2: (A.D. Aleksandrov) The following three conditions are equivalent.

(a) The sequence μ_1, μ_2, ... of measures in $\mathcal{M}$ converges weakly to μ^*;

(b) Lim sup $\mu_n(F) \leq \mu^*(F)$, for every closed set F. (27)

(c) Lim inf $\mu_n(G) \geq \mu^*(G)$, for every open set G. (28)

We are now ready to state our first existence theorem. For reasons which will become clear later, we treat the abstract transportation problem only for the case where total capacity equals total requirement. To assess the practical scope of this theorem, it should be noted that N-dimensional Euclidean space is

[8] P. Billingsley, Convergence of Probability Measures (N.Y., Wiley, 1968), Chapter I, is a very clear exposition of the theory for the special case of probabilities. K. R. Parasarathy, Probability Measures on Metric Spaces (N.Y.: Academic Press, 1967), is also useful.

separable and topologically complete; thus the following theorem and its generalizations would appear to cover almost all cases to be met with in practice.

Theorem 7: Let (A, Σ', μ'), (B, Σ'', μ''), and r be as in the abstract transportation problem, (4) - (6). In addition, assume that Σ' is the Borel field of a topology $\mathcal{J}'$ over A, which is separable and topologically complete; similarly, Σ'' is assumed to be the Borel field of a topology $\mathcal{J}''$ over B, which is separable and topologically complete. r is also assumed to be continuous with respect to the product topology $(A \times B, \mathcal{J}' \times \mathcal{J}'')$. Finally, assume $\mu'(A) = \mu''(B)$.

Then there exists an optimal solution ϕ^* to the abstract transportation problem.

Proof: There exists a feasible solution, by theorem 1. Also the set of values assumed by the objective function (6) is bounded, since r is bounded, and feasible ϕ's are bounded by constraint (4). Hence there exists a finite infimum, V, and a sequence of feasible flows ϕ_1, ϕ_2, ... such that

$$\lim_{n\to\infty} \int_{A\times B} r \, d\phi_n = V. \tag{29}$$

We will show that ϕ_1, ϕ_2, ... converges weakly to an optimal measure. It is known that, in a separable and topologically complete space, any measure on the Borel field is tight (Ulam's theorem; see Billingsley p. 5-6). Thus μ' and μ'' are tight. Hence, for all positive ϵ, there are compact sets $K' \subset A$, $K'' \subset B$, such that $\mu'(A \setminus K') < \epsilon$ and $\mu''(B \setminus K'') < \epsilon$.

Now $K' \times K''$ is compact in the product topology (Tihonov's theorem), and one verifies that, for any feasible ϕ,

$$\phi(A\times B \setminus K'\times K'') \leq \phi(A\times(B\setminus K'')) + \phi((A\setminus K')\times B) = \mu''(B\setminus K'') + \mu'(A\setminus K') < 2\epsilon \tag{30}$$

(The equality in (30) follows from the fact that all constraints in (4) and (5) are actually equalities here -- from theorem 2 above.) Hence the set of feasible solutions is uniformly tight. We now apply Prohorov's theorem (Lemma 1) to conclude that the set of feasible solutions is weakly relatively compact. (To justify this, we note that $\mathcal{T}' \times \mathcal{T}''$ is metrizable since its components are, and $\Sigma' \times \Sigma''$ is the Borel field of $\mathcal{T}' \times \mathcal{T}''$ since $\mathcal{T}'$ and $\mathcal{T}''$ are separable as well.)

Thus the sequence $\phi_1, \phi_2, \ldots$ contains a subsequence which converges weakly. For simplicity we use the same notation for this subsequence. It is clear that (29) still applies to the subsequence. Letting ϕ^* be the weak limit of the subsequence, we obtain

$$V = \lim_{n \to \infty} \int_{A \times B} r \, d\phi_n = \int_{A \times B} r \, d\phi^*, \tag{31}$$

from the definition of weak convergence. Thus ϕ^* attains the infimum, and we need only show that it is feasible to prove that it is optimal.

To prove feasibility, let ϕ'_n, ϕ''_n, and $\phi^{*\prime}$, $\phi^{*\prime\prime}$, be the left and right marginal measures of ϕ_n and ϕ^*, respectively ($n = 1, 2, \ldots$). The fact that $\phi_1, \phi_2, \ldots$ converges weakly to ϕ^*, implies that $\phi'_1, \phi'_2, \ldots$ converges weakly to $\phi^{*\prime}$, and similarly for the right-marginals. (Mann-Wald theorem; see Billingsley, p. 30-31.)

Now let G be any open subset of A. We obtain

$$\mu'(G) \geq \liminf \phi_n'(G) \geq \phi^{*\prime}(G). \tag{32}$$

(The first inequality comes from the fact that each measure $\phi_1, \phi_2, \ldots$ is feasible, hence satisfies (4); the second comes from Lemma 2.)

Similarly, let F be any closed subset of B. We obtain

$$\mu''(F) \leq \limsup \phi_n''(F) \leq \phi^{*\prime\prime}(F). \tag{33}$$

It is known that the measure of any Borel set of a metrizable topological space equals the infimum of the measures of the open sets containing it, and also equals the supremum of the measures of the closed sets contained in it. Let E be any Borel subset of A. We obtain

$\mu'(E) = \inf \{\mu'(G) | G \supset E, \text{ G open}\} \geq \inf \{\phi^{*\prime}(G) | G \supset E, \text{ G open}\} = \phi^{*\prime}(E)$ (34)

(The inequality comes from (32).) Also, if E is any Borel subset of E, we obtain

$\mu''(E) = \sup \{\mu''(F) | F \subset E, \text{ F closed}\} \leq \sup \{\phi^{*\prime\prime}(F) | F \subset E, \text{ F closed}\} = \phi^{*\prime\prime}(E)$. (35)

(The inequality comes from (33).)

ϕ^* satisfies the constraints (4), according to (34), and the constraints (5), according to (35). Hence it is feasible, hence optimal. QED

What happens if we allow $\mu'(A)$ to be greater than $\mu''(B)$? That is, we consider the more general case in which the capacity of sources can exceed the requirement of sinks. Surprisingly, theorem 7 breaks down:

<u>Theorem 8</u>: There is an abstract transportation problem satisfying all the premises of theorem 7, except that $\mu'(A) > \mu''(B)$, for which no optimal solution exists.

<u>Proof</u>: Let there be just one source, $\underline{a}$, of capacity 1. Let there be a countable number of sinks: $B = \{b_1, b_2, \ldots\}$ with requirements identically zero. Let every subset of B be open. Let the unit transport cost function be: $r(a, b_n) = \frac{1}{n} - 1$. One easily verifies that all the premises of theorem 7 hold (except, of course, that $\mu'(A) = 1$, $\mu''(B) = 0$). Yet there is no optimal flow, since if n is the smallest integer for which $\phi\{a, b_n\} > 0$, shifting this flow to the sink b_{n+1} reduces costs, while the identically zero flow is the worst of all. QED

There are, however, certain conditions under which we can still assert the existence of an optimal flow in the slack capacity case.

<u>Theorem 9</u>: Let the premises of theorem 7 be altered as follows.

(1) $\mu'(A) \geq \mu''(B)$ (replacing the stronger condition $\mu'(A) = \mu''(B)$)

(2) $(B, \mathcal{J}'')$ is compact and metrizable (replacing the weaker condition that $(B, \mathcal{J}'')$ is separable and topologically complete).

Then an optimal solution exists to the abstract transportation problem.

Proof: The place where the proof of theorem 7 breaks down if we replace $\mu'(A) = \mu''(B)$ by an inequality is in relation (30), where we cannot assert that

$$\emptyset(A \times (B\backslash K'')) + \emptyset((A\backslash K') \times B) = \mu''(B\backslash K'') + \mu'(A\backslash K'),$$

since theorem 2 does not apply. However, if B itself is compact, we can replace K'' by B, to obtain

$$\emptyset(A \times B\backslash K' \times B) = \emptyset((A\backslash K') \times B) \leq \mu'(A\backslash K') < \epsilon. \qquad (36)$$

Hence the set of feasible solutions is again uniformly tight, and the proof proceeds exactly as in theorem 7. QED

Theorem 10: Let the premises of theorem 7 be altered as follows: $\mu'(A) \geq \mu''(B)$, and r is non-negative.

Then an optimal solution exists to the abstract transportation problem.

We just outline the proof, rather than carrying out the somewhat tedious details. Given the original problem, we construct a new problem by adding an extra point (z) to the destination space B. We extend the topology $\mathcal{T}''$ by specifying that $G \cup (z)$ is to be open iff G is open in $\mathcal{T}''$. We extend the measure by specifying that the requirement for the singleton set (z) is to be $\mu'(A) - \mu''(B)$. Finally, we extend the function r by specifying that $r(x, z) = 0$ for all $x \in A$.

This new problem satisfies <u>all</u> the premises of theorem 7, <u>including</u> the condition that total capacity = total requirement. (Incidentally, this construction is the abstract form of the standard trick of adding an artificial sink to take up the slack in an inequality-constrained finite transportation problem.) Hence there exists an optimal flow $\emptyset^{**}$ for it. Let $\emptyset^*$ be this flow restricted to the original product space A x B. We claim that $\emptyset^*$ is optimal for the original problem.

To show this, let ϕ be any feasible flow for the original problem. Using the Radon-Nikodym theorem, we can show the existence of another feasible flow, $\tilde{\phi}$, which satisfies

$$\tilde{\phi} \leq \phi, \tag{37}$$

and satisfies the destination requirements exactly:

$$\tilde{\phi}(A \times E) = \mu''(E), \text{ for all } E \in \Sigma''. \tag{38}$$

It follows from (38) that $\tilde{\phi}$ has an extension, $\tilde{\tilde{\phi}}$, to the extended space $A \times (B \cup (z))$ which is feasible for the modified transportation problem. Finally, putting everything together, we obtain

$$\int_{A \times B} r \, d\phi^* = \int_{A\times(B\cup(z))} r \, d\phi^{**} \leq \int_{A\times(B\cup(z))} r \, d\tilde{\tilde{\phi}} = \int_{A\times B} r \, d\tilde{\phi} \leq \int_{A\times B} r \, d\phi, \tag{39}$$

which proves that ϕ^* is optimal.

(The equalities in (39) follow from the fact that $r(x, z) = 0$; the first inequality follows from the optimality of ϕ^{**} for the modified transportation problem; the last inequality follows from $r \geq 0$ and (37).) QED

Finally, we give two results which extend the scope of theorem 7 considerably.

<u>Theorem 11</u>: Let the premises of theorem 7 be weakened to read: $(A, \mathcal{T}')$ and $(B, \mathcal{T}'')$ are <u>Borel</u> <u>subsets</u> of separable and topologically complete spaces. Then there still exists an optimal solution.

<u>Proof</u>: The only place where topological completeness is used in the proof of theorem 7 is to imply that μ' and μ'' are tight measures. But any measure on a space satisfying the weakened premises above is necessarily tight. (See Parasarathy, p. 29-30.) QED

In practical terms, theorem 11 means that existence can be asserted not only when the origin-destination spaces, A and B, are Euclidean spaces of arbitrary finite dimension, but also when they are more or less arbitrary subsets.

<u>Theorem 12</u>: Let the premises of theorem 7 be satisfied. Let λ and ν be two

given measures on $(A \times B, \Sigma' \times \Sigma'')$, and let feasible flows be required to satisfy not only (4) and (5), but to lie between λ and ν:

$$\lambda(E) \leq \phi(E) \leq \nu(E), \text{ for all } E \in \Sigma' \times \Sigma''. \tag{40}$$

Then, <u>if</u> there exists a feasible flow satisfying (4), (5), and (40), there exists an optimal flow for these constraints.

<u>Proof</u>: The proof of theorem 7 applies without change, except that we must explicitly assume the existence of a feasible solution to get started, since theorem 1 is not available. Thus, we know there exists a sequence, $\phi_1, \phi_2, \ldots$ of feasible solutions which converges weakly to a measure ϕ^* satisfying (4) and (5), and which achieves the infimum of the objective function (6). It remains to show that ϕ^* satisfies (40). This can be done in exactly the same way that the satisfaction of (4) and (5) are proved; namely, we show that

$$\nu(G) \geq \phi^*(G) \tag{41}$$

for any open set $G \subset A \times B$, and that

$$\lambda(F) \leq \phi^*(F) \tag{42}$$

for any closed set $F \subset A \times B$, by the same arguments leading to (32) and (33). Then, since $\Sigma' \times \Sigma''$ is the Borel field of the metrizable space $(A \times B, \mathcal{J}' \times \mathcal{J}'')$, the inequalities (41) and (42) can be extended to all Borel sets of $A \times B$, by the arguments of (34) and (35). Thus (40) is satisfied, ϕ^* is feasible for the more restrictive problem, hence optimal for it. QED[9]

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[9] Also an unlimited number of further constraints of the form $\phi(G_i) \leq \alpha_i$ or $\phi(F_i) \geq \beta_i$ may be imposed, where G_i and F_i are open and closed sets, respectively, in $A \times B$. The proof that feasibility implies optimality is exactly as in theorem 12: just put the subscript i in lines (41) and (42).

6. Existence of Potentials

We have seen that, if ϕ and (p, q) are feasible for their respective problems, and (p, q) is a measure potential for ϕ, then ϕ is optimal. Here we want to tackle the converse (and much more difficult) problem. If ϕ is optimal, is there a dual-feasible pair of functions (p, q) which is a measure potential for ϕ?

Actually, as we have mentioned, our results do not produce measure potentials directly, but a related property called a topological potential. We proceed to define this.

Let the quadruple $(A, \mathcal{T}, \Sigma, \mu)$ be, respectively, a set A, a topology $\mathcal{T}$ over A, a sigma-field Σ over A, and a measure μ on Σ. (We do not necessarily assume that $\mathcal{T} \subset \Sigma$, as we have been doing up to now.) Set $E \subset A$ is a <u>neighborhood</u> of point $x \in A$ if there is an open set G such that $x \in G$ and $G \subset E$. x is a <u>point of support</u> of the measure μ if every measurable neighborhood of x has positive μ-measure. The set of all points of support is called the <u>support</u> of μ.

<u>Definition</u>: Let ϕ be feasible for the abstract transportation problem (4) - (5), and $(p, q) \geq 0$ feasible for the dual problem (15). (p, q) is a <u>topological potential</u> for ϕ if the following three conditions are satisfied:

If $(x, y) \in A \times B$ is a point of support of ϕ, then

$$q(y) - p(x) = r(x, y). \qquad (43)$$

If $x \in A$ is a point of support of $(\mu' - \phi')$, then

$$p(x) = 0. \qquad (44)$$

If $y \in B$ is a point of support of $(\phi'' - \mu'')$, then

$$q(y) = 0. \qquad (45)$$

In (43), (x, y) being a point of support of ϕ refers, of course, to the product space, so that the quadruple used in defining the concept would be

$(A \times B, \mathcal{J}' \times \mathcal{J}'', \Sigma' \times \Sigma'', \phi)$. In (44), ϕ' is the left-marginal measure of ϕ, so that $(\mu' - \phi')$ is the measure of unused capacity of the sources; the corresponding quadruple is $(A, \mathcal{J}', \Sigma', \mu' - \phi')$. In (45), $(\phi'' - \mu'')$ is the measure of the oversupply above requirements arriving at the sinks; the corresponding quadruple is $(B, \mathcal{J}'', \Sigma'', \phi'' - \mu'')$.

The same definition also serves for the pair of equality-constrained programs, except that (p, q) need not be non-negative. Note also that for equality constraints, (44) and (45) are automatically fulfilled, so that they may be omitted from the definition. (This follows from the fact that, for equality constraints, $(\mu' - \phi')$ and $(\phi'' - \mu'')$ are identically zero, and therefore have no points of support; (44) and (45) are thus vacuously true.)

(43) - (45) have as much claim to generalize the "complementary slackness" conditions of duality theory as do the corresponding conditions (23) - (25) for measure potentials. Indeed, all three concepts coincide for the finite case (with all subsets open and measurable). (x, y) being a point of support of ϕ generalizes the notion in the finite case that there is a positive flow from origin x to destination y. The complementary slackness condition requires in this case that the dual relation for the pair (x, y) be fulfilled with equality, and this is exactly what (43) requires. Again, if there is unused capacity at a source, the complementary slackness condition requires that the dual variable be zero, just as relation (44) does. Relation (45) is a generalization of the analogous condition for oversupplied sinks.

It is of interest to find conditions under which a topological potential will also be a measure potential, for this, combined with the other results of the present section, will guarantee that an optimal solution of the primal problem implies an optimal solution to the dual such that the two problems have the same value. We need the following topological concept.

Definition: A topological space has the strong Lindelöf property if, for every collection of open sets $\mathcal{G}$, there is a countable subcollection $\mathcal{G}' \subset \mathcal{G}$, such that $\cup\mathcal{G} = \cup\mathcal{G}'$.

Any subset of Euclidean N-space -- indeed, any separable metrizable space -- has the strong Lindelöf property, so that it includes most cases of practical interest. We now have

Theorem 13: If (p, q) is a topological potential for $\emptyset$, and the product space $(A \times B, \mathcal{T}' \times \mathcal{T}'')$ has the strong Lindelöf property, then (p, q) is a measure potential for $\emptyset$.

Proof: First we show that (43) implies (23). A point (x, y) such that $q(y) - p(x) < r(x, y)$ is not a point of support of $\emptyset$, according to (43); hence it has a measurable neighborhood $N_{(x, y)}$ of $\emptyset$-measure zero. There is an open set $G_{(x, y)}$ such that $(x, y) \in G_{(x, y)} \subset N_{(x, y)}$. Consider the collection, $\mathcal{G}$, of all these open sets, one for each point (x, y) for which the strict inequality holds: $q(y) - p(x) < r(x, y)$. By the strong Lindelöf property, there is a countable subcollection $\{G_1, G_2, \ldots\}$ whose union equals $\cup\mathcal{G}$. Let $\{N_1, N_2, \ldots\}$ be the neighborhoods in which these G-sets are respectively contained. We then have

$$\{(x, y) \mid q(y) - p(x) < r(x, y)\} \subset (\cup\mathcal{G}) = (G_1 \cup G_2 \cup \ldots) \subset (N_1 \cup N_2 \cup \ldots). \tag{46}$$

Hence

$$\emptyset \{(x, y) \mid q(y) - p(x) < r(x, y)\} \leq \emptyset(N_1 \cup N_2 \cup \ldots) \leq \emptyset(N_1) + \emptyset(N_2) + \ldots = 0, \tag{47}$$

and this yields (23).

For the equality constrained case the proof is completed, since (24) and (25) are automatically fulfilled. For the inequality-constrained case, we have $(p, q) \geq 0$, and we now show that (44) implies (24) and (45) implies (25).

First, it is easily verified that the component spaces $(A, \mathcal{T}')$ and $(B, \mathcal{T}'')$ inherit the strong Lindelöf property from the product space.

From this point, the proofs copy the above reasoning exactly. We consider the set of points x in A such that $p(x) > 0$, find a neighborhood of each one of $(\mu' - \phi')$-measure zero, duplicate the reasoning involving open sets, and conclude that

$$(\mu' - \phi')\{x/p(x) > 0\} = 0, \tag{48}$$

which is the same as (24). Similarly, starting from the points y in B such that $q(y) > 0$, we find neighborhoods of $(\phi'' - \mu'')$-measure zero, and conclude that

$$(\phi'' - \mu'')\{y|q(y) > 0\} = 0, \tag{49}$$

which is the same as (25). QED

A condition for the opposite implication to hold is easier to find and to prove:

<u>Theorem 14</u>: If (p, q) is a measure potential for ϕ, and each of the three sets: (1) $\{(x, y)|q(y) - p(x) < r(x, y)\}$; (2) $\{x/p(x) > 0\}$; (3) $\{y/q(y) > 0\}$ is <u>open</u> in its respective space (1) $(A \times B, \mathcal{J}' \times \mathcal{J}'')$, (2) $(A, \mathcal{J}')$, (3) $(B, \mathcal{J}'')$), then (p, q) is a topological potential for ϕ. (In the equality-constrained case, it is sufficient for the first set to be open.)

<u>Proof</u>: We will show that, if (43), (44), (45), respectively, is false, and the first, second, third set, respectively, is open, then (23), (24), (25), respectively, is false.

Suppose (43) is false. Then there is a point of support (x^o, y^o) of ϕ for which $q(y^o) - p(x^o) < r(x^o, y^o)$. The set $\{(x, y)/q(y) - p(x) < r(x, y)\}$, being open by assumption, is a neighborhood of (x^o, y^o); it is also measurable, hence it has positive ϕ-measure, which is to say that (23) is false.

This already proves the theorem for the equality-constrained case. For the inequality-constrained case, we have $(p, q) \geq 0$. The reasoning is the same:

Suppose (44) is false. Then there is a point of support x^o of $(\mu' - \phi')$ for which $p(x^o) > 0$. The set $\{x/p(x) > 0\}$ is measurable and open, hence

$$(\mu' - \phi')\{x/p(x) > 0\} > 0, \text{ contrary to (24).}$$

Similarly for (45) and (25). QED

Note that neither of these theorems makes any assumptions about the relations between the topologies and sigma-fields.

We now come to the main business of this section, which is to construct a topological potential associated with a given optimal solution to the abstract transportation problem. We shall concentrate on the equality-constrained case, which is somewhat easier to deal with, and indicate later what happens when we go to inequalities. The assumptions that have to be made to carry through the construction are quite moderate; the proofs, unfortunately, are rather long. We begin by proving a basic lemma, which is then applied to the proof of the main result.

Lemma 3: Let (A, Σ', μ'), (B, Σ'', μ'') and r be as in the transportation problem, (4) - (6), where, however, we insist that all constraints be satisfied with equality: (10), (11). Let ϕ be an optimal solution for this problem. Also assume that there are topologies $\mathcal{J}'$ over A, $\mathcal{J}''$ over B, such that $\mathcal{J}' \subset \Sigma'$, $\mathcal{J}'' \subset \Sigma''$, and r is continuous with respect to $\mathcal{J}' \times \mathcal{J}''$. (We need not assume that $\mathcal{J}' \times \mathcal{J}'' \subset \Sigma' \times \Sigma''$.)

Then, if $a_1, \ldots, a_n$ are any n points of A, $b_1, \ldots, b_n$ any n points of B (not necessarily distinct in either case), such that (a_i, b_i) is in the support of ϕ for $(i = 1, \ldots, n)$, it follows that

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n \leq a_1b_2 + a_2b_3 + \ldots + a_{n-1}b_n + a_nb_1. \qquad (50)$$

(where we have abbreviated $r(x, y)$ as xy, a notation we shall use throughout this section).

Proof: Choose a positive number ϵ. There are n open sets, $L_1, \ldots L_n$ in A, and n open sets $M_1, \ldots, M_n$ in B, with the following properties:

$a_i \varepsilon L_i$, $b_i \varepsilon M_i$ $(i = 1, \ldots, n)$,

$$xy > a_i b_i - \epsilon \text{ for all } x \varepsilon L_i,\ y \varepsilon M_i \quad (i = 1, \ldots, n), \tag{51}$$

$$xy < a_i b_{i+1} + \epsilon \text{ for all } x \varepsilon L_i,\ y \varepsilon M_{i+1} \quad (i = 1, \ldots, n) \tag{52}$$

(where we write M_1 for M_{n+1}, b_1 for b_{n+1})

To see this, note that, by the continuity of r, there are open sets about a_i and b_i such that (51) is satisfied, and open sets about a_i and b_{i+1} such that (52) is satisfied. This yields two open sets about each of the points $(a_1, \ldots, a_n \quad b_1, \ldots, b_n)$. The intersection of these two satisfy all conditions simultaneously.

$$\text{Let } C = \frac{1}{n} \operatorname*{Min}_i \left[\phi(L_i \times M_i) \right]. \tag{53}$$

C is positive, since $L_i \times M_i$ is a measurable neighborhood of (a_i, b_i), a point of support of ϕ.

We now alter the flow ϕ by <u>adding</u> to it n measures ϕ_i^* $(i = 1, \ldots, n)$ and <u>subtracting</u> from it another n measures ϕ_i^{**} $(i = 1, \ldots, n)$. These are defined as follows on measurable rectangles $E \times F$ $(E \varepsilon \Sigma', F \varepsilon \Sigma'')$.

$$\phi_i^{**}(E \times F) = \frac{C\, \phi\, (\, (E \times F) \cap (L_i \times M_i)\,)}{\phi(L_i \times M_i)}. \qquad (i = 1, \ldots, n) \tag{54}$$

$$\phi_i^*(E \times F) = \frac{C\, \phi\, (\, (E \cap L_i) \times M_i)\, \phi\, (L_{i+1} \times (F \cap M_{i+1})\,)}{\phi(L_i \times M_i)\, \phi(L_{i+1} \times M_{i+1})} \quad (i = 1, \ldots, n) \tag{55}$$

(1 is to be substituted for n + 1 in the formula for ϕ_n^*.)

It is easy to see that (54) defines a measure; it is, in fact, proportional to ϕ in the rectangle $L_i \times M_i$, and zero outside it. (55) is zero outside the rectangle $L_i \times M_{i+1}$, and is in "product-measure" form on the rectangle. A standard extension theorem assures us that it also extends to a measure on $\Sigma' \times \Sigma''$.

We claim that the altered flow, $\phi + \Sigma_{i=1}^{n} (\phi_i^* - \phi_i^{**})$, remains feasible for the transportation problem with equality constraints. First we note that

$$\phi_i^*(E \times B) = \phi_i^{**}(E \times B) = \frac{C\,\phi(\,(E \cap L_i) \times M_i)}{\phi(L_i \times M_i)}\,, \quad (i = 1, \ldots, n) \tag{56}$$

so that the marginal condition (10) remains satisfied. Also

$$\phi_{i-1}^*(A \times F) = \phi_i^{**}(A \times F) = \frac{C\,\phi(L_i \times (F \cap M_i)\,)}{\phi(L_i \times M_i)}\,, \quad (i = 1, \ldots, n) \tag{57}$$

(n is to be substituted for zero in $\phi_o^*\,(A \times F)$)

Adding up over all changes again leads to cancellation, so that the marginal conditions (11) remain satisfied.

It remains to show only that the altered flow is non-negative everywhere. The only negative summands appear on the rectangles $L_i \times M_i$. If the measurable set E is contained in a certain number of these rectangles, the quantity

$$C\,\phi(E)\,\Sigma \frac{1}{\phi(L_i \times M_i)} \tag{58}$$

is subtracted. (Here the summation extends over those i for which $G \subset L_i \times M_i$.) From the definition of C, this quantity cannot exceed $\phi(E)$, the original flow value. Hence non-negativity is preserved. This shows that the altered flow is feasible.

Since ϕ is optimal, the change in transportation costs induced by $\Sigma_{i=1}^{n}(\phi_i^* - \phi_i^{**})$ must be non-negative. Thus

$$\Sigma_{i=1}^{n} \int_{A \times B} r\, d\, \phi_i^* \geq \Sigma_{i=1}^{n} \int_{A \times B} r\, d\, \phi_i^{**}. \tag{59}$$

Now ϕ_i^* is zero outside the rectangle $L_i \times M_{i+1}$; on that rectangle, the inequality (52) applies; hence

$$\int_{A \times B} \left[a_i b_{i+1} + \epsilon\right] d\, \phi_i^* \geq \int_{A \times B} r\, d\, \phi_i^* \quad (i = 1, \ldots, n). \qquad (b_{n+1} \equiv b_1) \tag{60}$$

Similarly, $\phi_i{}^{**}$ is zero outside the rectangle $L_i \times M_i$; on that rectangle, the inequality (51) applies; hence

$$\int_{A\times B} \left[a_i b_i - \epsilon\right] d\, \phi_i{}^{**} \leq \int_{A\times B} r \, d\, \phi_i{}^{**} \qquad (i = 1, \ldots, n) \tag{61}$$

The left-hand integrands are merely constants; also $\phi_i{}^{*}(A \times B) = \phi_i{}^{**}(A \times B) = C$, for all $i = 1, \ldots, n$: hence, integrating out the constants, and putting (59), (60), and (61) together, we obtain

$$C(a_1 b_2 + a_2 b_3 + \ldots + a_n b_1) + n\, C \epsilon \geq C(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n) - n\, C \epsilon. \tag{62}$$

Since C is positive, and ϵ can be taken arbitrarily small, the basic inequality (50) is obtained. QED

Lemma 3 is actually a stronger result than would be obtained if we merely assumed ϕ to be optimal for the inequality-constrained problem (4) - (6). Indeed, suppose ϕ is optimal for the problem (4) - (6). Then it is necessarily also optimal for an equality-constrained subproblem, namely, the one in which its own marginals, ϕ' and ϕ'', play the roles of μ' and μ'', respectively. Therefore, inequality (50) holds for this ϕ.

We now come to the main result. The premises are the same as for Lemma 3. Functions (p, q) are constructed which together constitute a topological potential for ϕ. That is, they are bounded and measurable, satisfy the dual feasibility condition (15), and the topological potential condition (43). Note that these are a potential for the equality-constrained problem, and as such are not guaranteed to be non-negative. (44) and (45) are automatically satisfied by the fact that ϕ is an equality-constrained optimum. p and q are "constructed" in the sense that one can write an explicit formula for them.

First we need a few concepts relating to continuity. Let $(A, \mathcal{T})$ be a topological space, and let f be a real-valued function with domain A. f is said to be upper semi-continuous if every set of the form $\{x | f(x) < C\}$ is open (C being a real number); it is lower semi-continuous if every set of the form

$\{x|f(x) > C\}$ is open. If there is a sigma-field, Σ, over A such that $\mathcal{J} \subset \Sigma$, it follows at once from the definitions that every upper or lower semi-continuous function is measurable with respect to Σ. Let $\mathcal{F}$ be a bounded collection of real-valued functions, all with domain A; we define <u>inf $\mathcal{F}$</u> to be that function whose value at the point $x \in A$ is the infimum of the values assumed by the members of $\mathcal{F}$ at that point. <u>Sup $\mathcal{F}$</u> is defined analogously for the supremum. It is not hard to show that, if $\mathcal{F}$ is a collection of <u>continuous</u> functions, then inf $\mathcal{F}$ is upper semi-continuous and sup $\mathcal{F}$ is lower semi-continuous. Let $(A \times B, \mathcal{J}' \times \mathcal{J}'')$ be a product space, and let r be a real-valued function with domain A x B; r is <u>equi-continuous</u> if, for every positive number ϵ, and every $a \in A$, there is a set $G' \in \mathcal{J}'$ such that $a \in G'$, and $|r(x, y) - r(a, y)| < \epsilon$ for all $x \in G'$, $y \in B$, <u>and</u>, for every $\epsilon > 0$ and every $b \in B$, there is a set $G'' \in \mathcal{J}''$ such that $b \in G''$, and $|r(x, y) - r(x, b)| < \epsilon$ for all $x \in A$, $y \in G''$. Let (A, d) be a metric space, and let f be a function with domain A; f is <u>uniformly continuous</u> if, for all positive ϵ, there is a positive δ such that $d(x_1, x_2) < \delta$ implies that $|f(x_1) - f(x_2)| < \epsilon$. Let (A, d') and (B, d") be two metric spaces, and r a function with domain A x B; r is <u>uniformly continuous</u> if, for all positive ϵ, there is a positive δ such that $d'(x_1, x_2) < \delta$ and $d''(y_1, y_2) < \delta$ imply that $|r(x_1, y_1) - r(x_2, y_2)| < \epsilon$ ($x_1, x_2 \in A$, and $y_1, y_2 \in B$).

<u>Theorem 15</u>: Let (A, Σ', μ'), (B, Σ'', μ''), and r be as in the transportation problem with equality constraints, (10), (11), and (6). Let $\emptyset$ be an optimal solution for this problem. Assume that there are topologies $\mathcal{J}'$ over A, and $\mathcal{J}''$ over B, such that $\mathcal{J}' \subset \Sigma'$, $\mathcal{J}'' \subset \Sigma''$, and r is continuous with respect to $\mathcal{J}' \times \mathcal{J}''$.

Then there exist functions, p and q (with domains A and B, respectively) such that (p, q) is a topological potential for $\emptyset$; furthermore, p is lower and q is upper semi-continuous.

<u>Proof</u>: For any $a \in A$, we define $p(a)$ as follows:

(Let x stand for points of A, y for points of B, and abbreviate $r(x, y)$ as xy).

Consider the class of all <u>finite</u> sequences $(x_o, y_1, x_1, \ldots, y_n, x_n)$ beginning with $a = x_o$, having the property that (x_i, y_i) is a point of support for ϕ $(i = 1, \ldots, n)$. The <u>value</u> of this sequence is defined to be

$$-x_oy_1 + x_1y_1 - x_1y_2 + x_2y_2 - \ldots + x_ny_n. \tag{63}$$

(n is an arbitrary integer; we also allow the "sequence" consisting of x_o alone; this is assigned the value zero). $p(a)$ is now defined as the <u>supremum of the value of such permissible sequences beginning with a</u>.

Having defined p, we now define q as follows:

For any $b \in B$,

$$q(b) = \inf_{x \in A} \left[p(x) + x\,b\right] \tag{64}$$

We claim that the pair (p, q) is a topological potential for ϕ. First we show that p is <u>bounded</u>. Clearly $p \geq 0$, since the sequence consisting of x_o alone is permissible, and has value zero.

Let $(x_o, y_1, x_1, \ldots y_n, x_n)$ be a permissible sequence. According to (50) of Lemma 3,

$$0 \geq x_1y_1 - x_1y_2 + x_2y_2 - \ldots + x_ny_n - x_ny_1 \tag{65}$$

Adding $x_ny_1 - x_oy_1$ to both sides of (65), we get (63) on the right, so that the value of any permissible sequence is bounded above by $x_ny_1 - x_oy_1$. Let $M = \operatorname{Sup} |xy|$ over $x \in A$, $y \in B$; since r is bounded, M is finite. We have just shown that p is bounded. In fact

$$2M \geq p \geq 0. \tag{66}$$

It follows from this that q is bounded. In fact, from (64) and (66),

$$3M \geq q \geq -M. \tag{67}$$

Next we verify (15). In fact, $q(y) - p(x) \leq xy$ follows at once from the definition of q, (64).

Next we verify (43). Let (a, b) be a point of support for $\emptyset$. For any $x \varepsilon A$, we have

$$p(x) \geq - xb + ab + p(a). \tag{68}$$

To see this, note that the right-hand side of (68) is simply the supremum over all permissible sequences beginning (x, b, a ...); hence it cannot exceed p(x), which is the supremum over a wider class of permissible sequences. Hence $p(x) + xb$ attains its infimum at $x = a$. Therefore

$$q(b) = p(a) + ab, \tag{69}$$

so that (43) is verified.

It remains only to show that p and q are measurable with respect to their sigma-fields, Σ' and Σ'', respectively. We do this by proving the stronger result that p is lower and q upper semi-continuous.

Holding x fixed, and considering $p(x) + xy$ as a function of y, with domain B, we note that it is continuous with respect to $\mathcal{T}''$, since r is continuous with respect to $\mathcal{T}' \times \mathcal{T}''$.

$q = \inf \mathcal{F}$, where $\mathcal{F}$ is the collection of these functions for all possible values of $x \varepsilon A$; hence q is upper semi-continuous.

As for p, we first note that

$$p(x) \geq \sup_{y \varepsilon B} \left[q(y) - xy \right]. \tag{70}$$

This follows at once from the definition of q, (64). Now let x be a point for which $p(x) > 0$. For any positive $\in$, there must be a permissible sequence, beginning $(x, y_1, x_1, \ldots)$ whose value comes within $\in$ of p(x):

$$p(x) - \in < - xy_1 + x_1 y_1 + p(x_1) \tag{71}$$

Therefore

$$p(x) - \epsilon < q(y_1) - xy_1, \tag{72}$$

from the fact that (x_1, y_1) is a point of support of ϕ, together with (69).

From (70) and (72), and the fact that ϵ is arbitrary, we obtain

$$p(x) = \sup_{y \in B} \left[q(y) - xy\right], \tag{73}$$

whenever $p(x) > 0$. Therefore, we have in general

$$p(x) = \max \left[0, \sup_{y \in B} \left[q(y) - xy\right]\right]. \tag{74}$$

Holding y fixed, $q(y) - xy$, considered as a function of x with domain A, is continuous. Also the identically zero function is continuous. $p = \sup \mathcal{F}$, where $\mathcal{F}$ is now the collection of these functions for all possible values of $y \in B$, together with the identically zero function; hence p is lower semi-continuous.

QED

Theorem 16: If, in addition to the premises of theorem 15, r is equi-continuous, then there is a topological potential with p and q continuous (in their respective spaces $(A, \mathcal{J}')$ and $(B, \mathcal{J}'')$, of course).

If, in addition, d' on A x A, and d" on B x B are metrics such that r is uniformly continuous, then p and q are uniformly continuous.

Proof: We use the same construction as above, and show that it has these properties. Let r be equi-continuous. We show that q, defined by (64), is such that the set $\{y \mid \alpha < q(y) < \beta\}$ is open for all real numbers $\alpha < \beta$. Let q(b) lie in this set, and choose ϵ small enough so that

$$\alpha < q(b) - \epsilon < q(b) + \epsilon < \beta \tag{75}$$

There is an open neighborhood G" of b such that $|xy - xb| < \epsilon$ for all $x \in A$, $y \in G''$. It follows from (64) that $|q(y) - q(b)| \leq \epsilon$ for all $y \in G''$; hence, by (75), $G'' \subset \{y \mid \alpha < q(y) < \beta\}$; hence the latter set is open, so that q is continuous.

Analogous reasoning applies to the function

$$\tilde{p}(x) = \sup_{y \varepsilon B} \left[q(y) - xy \right], \tag{76}$$

which is therefore continuous; hence $p = \max(0, \tilde{p})$ is continuous.

To prove the second part, assume that r is uniformly continuous with respect to the metric spaces (A, d') and (B, d"). For any positive ϵ, there is a δ such that $d''(y_1, y_2) < \delta$ implies $|xy_1 - xy_2| < \frac{\epsilon}{2}$ for all $x \varepsilon A$; hence, again from (64), $|q(y_1) - q(y_2)| < \epsilon$ whenever $d''(y_1, y_2) < \delta$, so that q is uniformly continuous.

Analogous reasoning applies to (76), so that $\tilde{p}$ is uniformly continuous. Hence $p = \max(0, \tilde{p})$ is uniformly continuous. QED

Just as for Lemma 3, the conclusions of theorems 15 and 16 apply also if $\emptyset$ is optimal for the inequality-constrained transportation problem, since it is then also optimal an equality-constrained subproblem.

What has not been shown is that, in this case, there are functions p, q which are also non-negative and satisfy (44) and (45).[10]

We conclude with a theorem that wraps up several of our previous results.

Theorem 17: Let (A, Σ', μ'), (B, Σ'', μ'') and r be as in the abstract transportation problem with equality constraints. Let $\emptyset$ be optimal for the problem. In addition, suppose that one can find topologies $\mathcal{J}'$ over A and $\mathcal{J}''$ over B, such that (1) $\mathcal{J}' \subset \Sigma'$ and $\mathcal{J}'' \subset \Sigma''$; (2) $\mathcal{J}' \times \mathcal{J}''$ has the strong Lindelöf property; (3) r is continuous with respect to $\mathcal{J}' \times \mathcal{J}''$.

[10] Incomplete investigations make it likely that the construction in theorem 15 (or a slight modification of it, perhaps) will in fact have these properties as well, in the inequality-constrained case. At the present moment, however, this is conjectural.

Then there exist functions (p, q) with domains A, B, which are feasible and optimal for the dual problem, and for which the value of the dual equals the value of the primal.

Proof: (p, q) constructed in theorem 15 is a topological potential for $\emptyset$.

Hence, by the strong Lindelöf property and theorem 13, it is a measure potential. Hence, by theorem 6, the value of the dual equals the value of the primal, so that (p, q) is dual optimal. QED

One final comment. The fact that $\emptyset$ is optimal enters into the proof of theorem 15 in a tenuous fashion. It is used only to prove Lemma 3, and Lemma 3 is used only to prove that p is bounded. Hence, if $\emptyset$ is any feasible flow, not known to be optimal, and we carry out the construction of theorem 15, and it turns out to be bounded, then (p, q) is a topological potential for $\emptyset$. If, in addition, the strong Lindelöf property holds for $\mathcal{T}' \times \mathcal{T}''$, then the reasoning in the proof of theorem 17 assures us that $\emptyset$ is, in fact, optimal. Thus, in these circumstances, boundedness of (p, q) constructed in theorem 15 is both necessary and sufficient for optimality of $\emptyset$.

References

M. Beckmann, "A continuous model of transportation", Econometrica, 20: 643-660, October, 1952.

M. Beckmann, "The partial equilibrium of a continuous space market", Weltwirtschaftliches Archiv, 71: 73-87, 1953.

P. Billingsley, Convergence of Probability Measures, (New York: Wiley, 1968).

A. Faden, Foundations of Spatial Economics, (Baltimore: Johns Hopkins Press, forthcoming).

P. R. Halmos, Measure Theory, (Princeton: Van Nostrand, 1950).

L.V. Kantorovich, "On the translocation of masses", Management Science, 5: 1-4, October, 1958 (originally published in Doklady Nauk USSR, 37: #7-8, 199-201, 1942).

A. Orden, "The transhipment problem", Management Science, 2: 276-285, April, 1956.

K.R. Parasarathy, Probability Measures on Metric Spaces, (New York: Academic Press, 1967).